ENI6MA: Emergent Holographic Proofs-of-Knowledge

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Abstract

This document specifies an interactive, human-in-the-loop proof-of-knowledge protocol that projects multiple Agent-committed alphabets into a rotating, foliated manifold of hyperplanes. A Verifying Circuit derives per-ring spin states (offsets) from a committed entropy integer and an optional time coefficient, rotates the rings, and partitions them into n hyperplanes each round. The Agent selects a zone via a private morphism from UI inputs to zone indices; the corresponding multi-alphabet witness is captured and logged with auditable indices.

The protocol is fully configurable at commitment: the number of alphabets M, their contents and sizes, the hyperplane count 4, and the secret length L are agent-tunable. Secrets are strictly case-sensitive and *n-ary*: L may be any positive integer derived from the Agent's committed alphabets and application needs. While examples use $n = 6$ and $L = 6$ for clarity, the `PoK` mechanics apply to arbitrary n and L.

Entropy drives deterministic yet unpredictable per-round offsets via a Rosario modulo index function. Optional time mixing injects microsecond-resolution variability without sacrificing reproducibility given the same $(, \tau)$. The Verifying Circuit alone computes offsets and foliations; the Agent operates only as a projective interface and submission layer, preserving clean security separation.

Each round produces n witnesses (one per zone) as concatenations of selected slices from the rotated alphabets. The Agent's private morphism φ maps inputs to zones, compounding adversarial uncertainty by n! morphism configurations. The Verifying Circuit logs indices and offsets to enable deterministic replay and external audit.

The acceptance rule is conjunctive across the secret's symbols: with $T \geq L$ rounds, every symbol s_i must be found in the corresponding witness X_i. This is equivalent to an accumulator Λ formed from membership predicates, yielding a clear `PASS/FAIL` outcome with no normalization and strict case sensitivity.

The framework generalizes across modalities (text, emoji, images, tones, haptics) and scales security via alphabet sizes, number of alphabets, hyperplane

count n, and entropy/time policies. It is designed for practical deployments that require human-centric interaction, deterministic auditability, and strong resistance to replay and forgery.

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0) Notation and Objects

Chapter Overview

This foundational section establishes the mathematical framework and symbolic language that underlies the entire Rosario-Wang cryptographic architecture. The notation and objects defined here serve as the building blocks for all subsequent cryptographic operations, providing the formal mathematical foundation necessary for rigorous security analysis and implementation. The section introduces the concept of arbitrary enumerated alphabets that can span multiple modalities, from traditional text-based character sets to more complex symbolic representations including emoji, images, audio tones, and haptic feedback patterns.

The importance of this section cannot be overstated, as it defines the fundamental objects that will be manipulated throughout the proof-of-knowledge

protocol. By establishing a flexible alphabet system that supports arbitrary cardinalities and modalities, the architecture enables unprecedented adaptability in cryptographic applications while maintaining mathematical rigor. The notation system provides a unified language for describing complex cryptographic operations across diverse implementation contexts, from embedded systems to cloud platforms.

The interoperation with the manifold cypher and proof system is direct and essential: the alphabets defined here become the rings that are rotated, foliated, and projected into hyperplanes during the manifold construction phase. The case sensitivity requirements and concatenation operators establish the mathematical rules that govern how these alphabets interact and combine to form witnesses. The zone enumeration system creates the spatial framework that enables the hyperplane projection and witness selection mechanisms.

The sub-sections systematically build from basic alphabet definitions to complex multi-modal configurations. Section 1.A introduces the core concept of arbitrary enumerated alphabets and their tunability, establishing the foundation for the entire system. Section 1.B defines the zone/color system that enables spatial organization of the cryptographic space. Section 1.C establishes the concatenation operator that combines alphabet slices into composite witnesses. Finally, Section 1.D emphasizes the critical importance of case sensitivity in maintaining cryptographic strength and preventing information leakage through normalization.

0.A) Alphabets (Agent-tunable; arbitrary enumerated rings across modalities)

- Arbitrary enumerated alphabets: the Agent may choose any separable, cardinally symmetric enumerated sets (e.g., Latin letters, digits, emoji, icons, images, audio tones, haptics/vibrations, magnetic or electronic pulses, or other eigenvalues). Ring sizes are arbitrary and fixed by commitment; optional balancing symbols (e.g., ♠♣) may be included by the Agent solely to achieve desired cardinal symmetry. In large scripts (e.g., Han Chinese), balancing symbols are typically unnecessary.

- Example (Latin default) — Upper: S_U = "ABCDEFGHIJKLMN♠OPQRSTUVWXYZ♣" (example size $|S_U| = 30$). Actual uppercase ring content/size is Agent-chosen and arbitrary.

- Example (Latin default) — Lower: S_L = "abcdefghijklmn♠opqrstuvwxyz♣" (example size $|S_L| = 30$). Agents may choose distinct lower rings or omit lowercase entirely.

- Example (Latin default) — Numeric: S_N = "12345*67890#" (example size $|S_N| = 12$). Numeric rings, and any additional rings, can be larger or smaller per commitment.

0.B) Zones / Colors

Indices $j \in \{0, 1, 2, 3, 4, 5\}$ map to agent-specified color enumerations

- The specific color mapping (e.g., $0 \to$ BLUE, $1 \to$ YELLOW, $2 \to$ RED, $3 \to$ WHITE, $4 \to$ GREEN, $5 \to$ BLACK) is determined by the authenticating Agent during the commitment phase

- The six zones represent distinct color-coded regions in the agent projective interface, each corresponding to a specific slice index in the 6-way foliation. This color mapping provides an intuitive visual representation of the mathematical zones, allowing authenticating agents to make selections based on visual cues rather than abstract indices. The color scheme is designed for high contrast and accessibility, ensuring that authenticating agents can easily distinguish between different zones during the interactive selection process.

  Each zone index j serves as a mathematical identifier that determines which slice of each alphabet (upper, lower, numeric) will be combined to form the witness for that round. This mapping creates a direct correspondence between agent projective interface elements and mathematical operations, bridging the gap between human interaction and algorithmic computation.

0.C) Concatenation Operator

Concatenation is denoted by $\circ$. The concatenation operator $\circ$ represents the joining of character sequences end-to-end. This operation is fundamental to the witness construction process, where slices from different alphabets are combined to create composite witnesses. The concatenation preserves the order and content of each slice while creating a unified character set that represents the full character space available in a particular zone.

0.D) Case Sensitivity

Witnesses, secrets, and verification are strictly case-sensitive; no normalization (e.g., lowercasing) is applied. Case sensitivity increases the permutation space and strengthens resistance to brute force.

- **Entropy (configurable, integer 256–512 bits)**
  Let $\mathcal{E} \in \mathbb{N}$ be a base entropy integer selected during the commitment phase. Its size is arbitrary (commonly 256–512 bits). This integer drives character ring rotation offsets via a Rosario modulo index function:

  - Fractal reduction (Rosario modulo index): for each alphabet/ring α with size $|\alpha|$, an offset is obtained by reducing an entropy-derived block modulo $|\alpha|$; nested or repeated reductions can be used to influence multiple dimensions.

– Time mixing (optional): a time coefficient τ (e.g., Unix-epoch microseconds) may be multiplied into entropy slices before reduction to yield per-μs variation.

– Purpose: the resulting offsets Θ_α are applied as Möbius rotation amounts during the manifold's stochastic construction stage, prior to foliation into hyperplanes.
See Section 3 for a comprehensive description and equations.

0.E) Symbols and Parameters (global)

- **M**: number of distinct alphabets/rings (Agent-tunable; $M \geq 1$)
- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_M\}$: ordered family of alphabets (rings); $|\alpha_a|$ is the cardinality of ring a
- **n**: number of hyperplanes/zones (Agent-tunable; $n \geq 2$); zone index set $\mathcal{Z} = \{0, 1, \ldots, n-1\}$ (0-based)
- **L**: secret length; secret string $s = s_1 \cdots s_L$ with case-sensitive symbols from $\bigcup_{a=1}^{M} \alpha_a$
- **T**: number of completed rounds in a transcript (must satisfy $T \geq L$ for acceptance)
- $\boldsymbol{\mathcal{E}} \in \mathbb{N}$: base entropy integer; $|\mathcal{E}|$ denotes bit length (commonly 256–512)
- $\boldsymbol{\tau} \in \mathbb{N}$: optional microsecond time coefficient used for time mixing
- **U**: usable slice window size for human-auditable entropy slicing (default $U = 25$)
- $\mathbf{e_i}$: base-10^3 entropy slices of $\mathcal{E}$; $e_i \in \{0, \ldots, 999\}$
- $\tilde{e}_i$: time-mixed slices, $\tilde{e}_i = (\tau \cdot e_i) \bmod 10^3$
- $\mathbf{q}, \mathbf{r}$: base slice length and remainder for n-way foliation, $q = \lfloor |s|/n \rfloor$, $r = |s| \bmod n$
- $\boldsymbol{\ell_j}$: per-zone slice lengths, $\ell_j = q + \mathbf{1}\{j < r\}$; prefix indices $a_0 = 0$, $a_{j+1} = a_j + \ell_j$
- $\mathbf{q}_\alpha$: ring-specific base lengths for foliation, $q_\alpha = \lfloor |\alpha|/n \rfloor$ (used in logged indices)
- $\mathbf{k}_\alpha^{\mathbf{t}}$: per-round, per-ring rotation seeds (when sampling with RNG), with $\alpha \in \{U, L, N, \text{or general}\}$ and round $t \in \{1, \ldots, T\}$

Indexing conventions: zone indices j are 0-based in $\{0, \ldots, n-1\}$; rounds may be denoted by $t \in \{1, \ldots, T\}$ or r; both appear in later sections. String positions are 0-based where used in rotation and slicing.

0.F) Core Operators and Functions

- Möbius rotation ρ_k: for string s, $\rho_k(s) = s[k : |s| - 1] \circ s[0 : k - 1]$, $0 \le k < |s|$ ([E1])
- n-way foliation Π_n: returns $(s_0, \ldots, s_{n-1})$ with slice lengths ℓ_j and boundaries defined by (q, r, ℓ_j, a_j) ([E2])
- Index progression κ: $\kappa(s, n, k, j) = (k + j\,q) \bmod |s|$ records the origin index of slice j ([E3])
- Rosario block selection $\kappa_{r,j}$: $\kappa_{r,j} = ((\tau \bmod U) + r\,M + j) \bmod U$; selected block $\text{block}_{r,j} = \tilde{e}_{1+\kappa_{r,j}}$ ([E8])
- Offset formation $\Theta^{(r)}_{\alpha_j}$: $\Theta^{(r)}_{\alpha_j} = \text{block}_{r,j} \bmod |\alpha_j|$; rotated ring $(\alpha_j)' = \rho_{\Theta^{(r)}_{\alpha_j}}(\alpha_j)$
- Concatenation $\circ$: joins strings end-to-end (see §1.C)
- Indicator $\mathbf{1}\{\cdot\}$: equals 1 if predicate holds, else 0
- Membership predicate $\mathrm{M}(p, x)$: true iff symbol p occurs in string/subset x
- Zone morphisms:
  - Default mapping m: concrete key-to-zone map (example in §6.B)
  - Reversed mapping m_r: alternate key-to-zone map (example in §6.B)
  - General morphism $\phi : \mathcal{I} \to \mathcal{Z}$: private, bijective map from UI input space $\mathcal{I}$ to zone set $\mathcal{Z}$; m, m_r are concrete instances

0.G) Derived Objects (per round)

- Rotated alphabets: $\hat{U}^t = \rho_{k^t_U}(S_U)$, $\hat{L}^t = \rho_{k^t_L}(S_L)$, $\hat{N}^t = \rho_{k^t_N}(S_N)$ (or generally for $\alpha \in \boldsymbol{\alpha}$)
- Foliated slices: $(U^t_0, \ldots, U^t_{n-1}) = \Pi_n(\hat{U}^t)$, similarly for L and N
- Zone-j witness: $W^t_j = U^t_j \circ L^t_j \circ N^t_j$; in general, $W^t_j = \bigcirc_{a=1}^{M} \alpha^t_{a,j}$ ([E4], [E9])
- Zone selection: with input key k_t, $z_t = m(k_t)$ if DIRECTION_SWITCH=false, else $z_t = m_r(k_t)$; abstractly $z_t = \phi(k_t)$ ([E5])
- Captured witness and indices: $X_t = W^t_{z_t}$, $I_t = (\kappa(S_U, n, k^t_U, z_t),\ \kappa(S_L, n, k^t_L, z_t),\ \kappa(S_N, n, k^t_N, z_t))$ ([E6])
- Logged indices (general n): $I^{(\text{gen})}_t = ((k^t_U + q_U z_t) \bmod |S_U|,\ (k^t_L + q_L z_t) \bmod |S_L|,\ (k^t_N + q_N z_t) \bmod |S_N|)$ with $q_\alpha = \lfloor |\alpha|/n \rfloor$
- Transcript element: $T_t = (X_t, I_t)$; full transcript $\mathcal{T} = \{T_1, \ldots, T_T\}$

0.H) Verification and Acceptance Constructs

- Acceptance condition ([E7]):

$$\text{ACCEPT} \iff T \geq L \ \wedge \ \bigwedge_{i=1}^{L} \mathbf{1}\{\, s_i \in \text{chars}(X_i) \,\} = 1$$

- Acceptance indicator: $A(s, X^{1:T}) = \mathbf{1}\{T \geq L\} \prod_{i=1}^{L} \mathbf{1}\{s_i \in X_i\}$
- Accumulator (Rosario–Wang) ([11.I]): $\Lambda = \bigwedge_{R=1}^{n} \ \mathrm{M}(p_i, x_i^R)$; in the minimal 1-to-1 model $i = R$, $\Lambda = \bigwedge_{i=1}^{L} \mathrm{M}(s_i, X_i)$
- Decision: return PASS iff $A = 1$ (equivalently $\Lambda = 1$), else FAIL

0.I) Mathematical and Implementation Conventions

- Case sensitivity is strict across alphabets, witnesses, secrets, and verification; no normalization is permitted
- All modular arithmetic is taken over the natural range of the underlying string/ring length, e.g., mod$|s|$ or mod$|\alpha|$
- Unless stated, zone indices j use 0-based enumeration; round counters may appear as t or r and range from 1 to T
- Seeds k_α^t are generated independently per alphabet and per round; when entropy/time is used, offsets are derived deterministically from $(\mathcal{E}, \tau)$ via $\kappa_{r,j}$ and $\Theta_{\alpha_j}^{(r)}$
- Logging captures both X_t and I_t for reproducibility and external audit; optional integrity protection can sign I_t
- Complexity: verification runs in $O(L)$ on the transcript $(X_1, \ldots, X_T)$ with simple membership checks

```
classDiagram
  class Alphabet {
    +name
    +size |α|
    +content
  }
  class Zone {
    +index j
    +color
  }
  class Secret {
    +s : string
    +L : length
```

```
}
class Witness {
  +X_t : string
  +z_t : zone
}
class Transcript {
  +T : list of (X_t, I_t)
}
Alphabet <.. Witness : "slices used"
Zone o-- Witness : "selects j"
Secret --> Transcript : "verified across rounds"
Transcript *-- Witness : "collects"
```

1) Commitment Phase

The Commitment Phase is the foundation of the Rosario-Wang cryptographic architecture, where authenticating agents establish their cryptographic identity and commit to parameters that govern all subsequent operations. This phase creates the essential foundation for the proof-of-knowledge protocol and manifold cypher system.

During the commitment phase, agents establish cryptographic components including arbitrary enumerated alphabets as foundational rings, secret key arrays derived from these alphabets, and private witness maps that establish bijective relationships between enumerated alphabets. These components create a multi-layered security framework.

The commitment process compiles these parameters into a secure binary format. It begins with local substitution alphabets that maintain cardinal symmetry and bijective properties. The process then encrypts these commitments into a binary executable, with options for encrypted sidecar files.

The resulting binary serves as the cryptographic engine that orchestrates all operations. It protects embedded secrets while providing the computational framework for the proof-of-knowledge protocol. The binary format ensures commitments are embedded within machine code rather than stored as accessible data structures.

Once compiled, commitments become permanently embedded within the machine code, making reverse engineering virtually impossible. The secrets are distributed throughout the executable's instruction stream, integrated with the machine code itself rather than stored as discrete data elements.

This architecture provides protection that traditional cryptographic key storage cannot match. Even with binary access, the distributed nature of embedded secrets makes reconstruction extremely difficult. It protects against both static and dynamic analysis attacks.

The commitment phase establishes the foundation for the manifold cypher system's deterministic behavior. Committed parameters serve as mathematical seeds that drive manifold construction, ensuring consistent and verifiable results while maintaining security properties. This deterministic behavior enables

verification without access to original secret parameters.

```
graph LR
  A["Agent"] --> P["Select parameters:<br/>M, n, L, φ, ||, τ_enabled"]
  A --> S["Choose alphabets {α..α_M}"]
  P --> C["Commitment tuple  = (M,{α},n,L,φ,||,τ_enabled)"]
  S --> C
  C --> B["Compile/Embed into binary"]
  B --> O["Binary Engine (with embedded commitments)"]
  B --> SC["Optional Config Sidecar"]
  O --> V["Verifying Circuit"]
```

1.A) Parameter Establishment

Before the interactive proof-of-knowledge protocol begins, the authenticating Agent engages in a commitment phase where critical parameters are established:

Values collected during the commitment phase by the authenticating agent:

- **Alphabets that are used for the secret key array**: The specific character sets (e.g., S_U, S_L, S_N) that will be employed in the protocol
- **The secret key array derived from these alphabets**: The actual secret string $s = s_1 s_2 \cdots s_L$ that the authenticating agent must prove knowledge of. The secret length L is n-ary and agent-tunable (arbitrary $L \geq 1$) derived from the committed alphabets and application requirements; examples in this document may use $L = 6$ for clarity only.
- **A private witness map, derived from enumerated alphabets that are BIJECTIVE and cardinally symmetric**: A mapping between two cardinally symmetric enumerated alphabets for the projection and for the witnesses

1.B) Color/Zone Enumeration Specification

The authenticating Agent specifies an arbitrary number of hyperplanes and a mapping between enumerated alphabets. The number of colors (four or six: blue, yellow, red, green, white, black) are simply agent-selected enumerations specified during the commitment phase.

Example mapping specification:

- UP equals green
- Down equals yellow
- Forward equals black
- etc.

This mapping creates a bijective relationship between agent projective interface elements and mathematical zone indices, ensuring that each input action corresponds to exactly one zone selection.

1.C) Roles: Authenticating Agent vs Verifying Circuit

- Authenticating Agent: selects and commits parameters (number of alphabets M, the enumerated alphabets themselves, hyperplane count n, secret length L, UI glyph mappings/morphisms), presents the UI, captures authenticating agent key inputs, and submits witness declarations. The Agent does not compute manifold ring offsets.

- Verifying Circuit: given the committed entropy ($\mathcal{E}$) and optional time coefficient (τ), constructs the manifold by slicing, time-mixing, selecting blocks via κ, computing per-ring offsets Θ, rotating rings, and foliating into hyperplanes. It logs indices and verifies witness membership. All manifold ring offsets and hyperplane constructions are computed by the Verifying Circuit, not the Agent.

- Security separation: this role split keeps sensitive offset construction and replayable determinism within the Verifying Circuit while allowing the Agent to remain an projective interface and submission layer only.

2) Entropy as Base Integer and Fractal Offset Application (Rosario Modulo Index)

Chapter Overview

This section introduces the revolutionary entropy-driven approach that distinguishes the Rosario-Wang architecture from traditional cryptographic systems. The section establishes how large base entropy integers (256-512 bits) serve as the foundational source of randomness for all cryptographic operations, replacing traditional pseudo-random number generation with true entropy-driven spin states. The fractal offset application through the Rosario modulo index function creates a deterministic yet unpredictable mapping from entropy blocks to ring-specific rotation offsets, enabling reproducible verification while maintaining the security properties of random sampling.

The importance of this entropy-driven approach lies in its ability to provide true randomness that cannot be predicted or manipulated by any party, while simultaneously enabling deterministic replay and verification. By using cryptographically strong entropy sources as the foundation for deriving all rotation offsets, the system achieves unprecedented security properties that scale exponentially with entropy size. The time mixing component adds a temporal dimension that makes the system resistant to replay attacks and ensures that each interaction produces a unique manifold even with identical entropy values.

The interoperation with the manifold cypher and proof system is fundamental: the entropy-derived offsets function as per-ring spin states that fully

determine the manifold construction at each round. These offsets drive the Möbius rotation operations that randomize alphabet ordering, and the resulting rotated rings are then foliated into hyperplanes to create the cryptographic space for witness generation. The entropy serves as the mathematical seed that coordinates all subsequent cryptographic operations, ensuring consistency and verifiability across the entire protocol.

The sub-sections systematically develop the entropy-driven architecture from basic principles to sophisticated implementation details. Section 2.A establishes the base entropy concept and its tunability across different security requirements. Section 2.B introduces the human-auditable slicing mechanism that makes entropy operations transparent and verifiable. Section 2.C describes the optional time mixing that adds temporal unpredictability. Section 2.D presents the core Rosario modulo index function that deterministically selects entropy blocks for each round and ring. Finally, Section 2.E explains how fractal reduction creates ring-specific offsets that drive the manifold construction process.

2.A) Base Entropy and Tunability

$\mathcal{E} \in \mathbb{N}$ is an Agent-chosen integer (commonly 256–512 bits). The Agent also selects the number of alphabets $M \geq 1$, their contents $\{\alpha_1, \ldots, \alpha_M\}$, the number of hyperplanes $n \geq 2$, and the secret length $L \geq 1$ during the commitment phase. Defaults in this document use $M = 3$ and $n = 6$, but all are tunable.

2.B) Human-Auditable Slicing

Write $\mathcal{E}$ in base 10^3 as 3-digit slices $e_i \in \{0, \ldots, 999\}$ and optionally reserve the head slice for guards; let the usable window size be U (default $U = 25$) as described in ENTROPY/THEORY.md.

2.C) Optional Time Mixing

Introduce a microsecond time coefficient τ and define time-mixed slices

$$\tilde{e}_i = (\tau \cdot e_i) \bmod 10^3,$$

keeping values within three digits while coupling manifold state to time.

2.D) Per-Round, Per-Ring Selection (Rosario Modulo Index)

For round r and ring coordinate $j \in \{0, \ldots, M-1\}$, deterministically select a slice index

$$\kappa_{r,j} = ((\tau \bmod U) + r\,M + j) \bmod U, \qquad \text{block}_{r,j} = \tilde{e}_{1+\kappa_{r,j}}.$$

Other bijective enumerations of the usable window are permitted; the key property is determinism given $(\mathcal{E}, \tau, r, j)$.

2.E) Fractal Reduction to Ring Offsets

For each selected alphabet α_j with size $|\alpha_j|$, compute the offset and apply Möbius rotation

$$\Theta_{\alpha_j}^{(r)} = \text{block}_{r,j} \bmod |\alpha_j|, \qquad (\alpha_j)' = \rho_{\Theta_{\alpha_j}^{(r)}}(\alpha_j).$$

When sub-rings or nested constructions are used, repeated ("fractal") reductions of the same $\text{block}_{r,j}$ against different moduli can derive families of offsets across dimensions.

- Role in manifold construction: these Θ offsets are applied during the manifold's stochastic construction stage to each character ring prior to n-way foliation into hyperplanes. With τ disabled the process is deterministic per session; with τ enabled, offsets refresh at microsecond cadence.
- Determinism and tunability: fixing $(\mathcal{E}, \tau)$ and the Agent's commitments $(M, \{\alpha\}, n, L)$ fully determines the per-round offsets and hence the emergent hyperplanes, while preserving per-μs variability when time mixing is enabled. See ENTROPY/THEORY.md §§1–7 for a deeper treatment including routing and complexity bounds.

```
graph TD
  E["Base entropy "] --> SL["Slice to base 10^3<br/>e_i  {0..999}"]
  SL --> TM["Optional time mixing τ:<br/>ė_i = (τ·e_i) mod 10^3"]
  SL -->|no τ| KP["_{r,j} selection"]
  TM --> KP["_{r,j} = ((τ mod U) + r·M + j) mod U"]
  KP --> BL["block_{r,j} = ė_{1+_{r,j}}"]
  BL --> TH["_{α_j}^{(r)} = block mod |α_j|"]
  TH --> RO["Rotate ring _(α_j)"]
```

3) Entropy-Driven Offsets and Sampling (Rosario Modulo Index)

Chapter Overview

This section provides a comprehensive treatment of the entropy-driven offset generation and sampling mechanisms that form the mathematical core of the Rosario-Wang cryptographic architecture. Building upon the foundational concepts introduced in Section 2, this section delves deeper into the practical implementation details of how entropy is transformed into actionable cryptographic parameters. The section establishes the complete mathematical framework for entropy processing, from initial slicing and time mixing to the final generation of per-round, per-ring rotation offsets that drive the manifold construction process.

The importance of this section lies in its detailed exposition of the entropy processing pipeline that transforms raw entropy into the precise mathematical parameters needed for cryptographic operations. By providing a complete mathematical description of the offset generation process, this section enables

implementers to create robust, verifiable systems that maintain the security properties of the original entropy while enabling efficient computation. The section also establishes the mathematical relationships between different entropy processing stages, ensuring consistency and correctness across the entire system.

The interoperation with the manifold cypher and proof system is operational: the offsets generated through the entropy processing pipeline become the exact rotation amounts applied to each alphabet ring during manifold construction. These offsets determine the starting positions for the foliation process that creates hyperplanes, and they establish the mathematical relationships that enable deterministic replay and verification. The entropy processing ensures that each round produces a unique, unpredictable manifold while maintaining the mathematical consistency necessary for cryptographic verification.

The sub-sections provide a systematic progression through the complete entropy processing pipeline. Section 3.A establishes the base entropy concept and its range specifications for different security levels. Section 3.B describes the default slicing mechanism that divides entropy into manageable blocks for processing. Section 3.C introduces the optional time mixing that adds temporal unpredictability to the entropy processing. Section 3.D presents the core Rosario modulo index function that deterministically selects entropy blocks for each round and ring combination. Section 3.E explains the offset formation process through fractal reduction that creates ring-specific rotation amounts. Finally, Section 3.F discusses the configurability aspects that allow the system to adapt to different deployment scenarios.

3.A) Base Entropy and Range

$\mathcal{E} \in \mathbb{N}$ (agent-chosen, e.g., 256–512 bits). Renderable to decimal for human-auditable slicing (e.g., 78–155 digits), but any fixed radix can be used.

3.B) Slicing (Default)

Write $\mathcal{E}$ in base 10^3 as 3-digit slices $e_i \in \{0, \ldots, 999\}$; optionally reserve a leading slice for guards/headers; usable window size is application-defined (default 25).

3.C) Time Mixing (Optional)

$\tilde{e}_i = (\tau \cdot e_i) \bmod 10^3$ to maintain three digits while coupling to time.

3.D) Per-Round, Per-Ring Selection (Rosario Modulo Index)

For round r and ring index j among M alphabets,

$$\kappa_{r,j} = ((\tau \bmod U) + r\,M + j) \bmod U, \qquad \text{block}_{r,j} = \tilde{e}_{1+\kappa_{r,j}},$$

where U is the usable slice count. Other bijective sampling schemes over the slice window are permitted; the essential property is deterministic selection per

$(\mathcal{E}, \tau, r, j)$.

3.E) Offset Formation (Fractal Reduction)

For each selected alphabet α_j with size $|\alpha_j|$, define

$$\Theta_{\alpha_j}^{(r)} = \text{block}_{r,j} \bmod |\alpha_j|,$$

and rotate the ring by this Möbius amount: $(\alpha_j)' = \rho_{\Theta_{\alpha_j}^{(r)}}(\alpha_j)$. Nested reductions can be applied to derive families of offsets when multiple rings or sub-rings are present.

3.F) Configurability

The number of alphabets M, their contents $\{\alpha_1, \ldots, \alpha_M\}$, the hyperplane count n, and the secret length L are agent-tunable during the commitment phase. Defaults in this document use $M = 3$ (uppercase/lowercase/numeric) and $n = 6$.

This section summarizes how $\mathcal{E}$ (and optionally τ) deterministically induces rotation offsets feeding the manifold projection. See ENTROPY/THEORY.md §§2–7 for the full theoretical framework. Offsets function as per-ring spin states; the Verifying Circuit, armed with $(\mathcal{E}, \tau)$, deterministically regenerates and, when needed, inverts them to evaluate witness membership.

```
sequenceDiagram
  autonumber
  participant E as "Entropy/Time"
  participant S as "Sampler "
  participant R as "Ring α_j"
  loop "For each round r and ring j"
    E->>S: "ė_i (slices)"
    S->>S: "_{r,j} over window U"
    S->>R: "_{α_j}^{(r)} = block mod |α_j|"
    R-->>S: "_(α_j)"
  end
```

4) Manifold Projection (Rotation + Foliation)

Chapter Overview

This section introduces the geometric foundation of the Rosario-Wang cryptographic architecture through the mathematical operations of rotation and foliation that transform abstract entropy-driven offsets into concrete, verifiable cryptographic structures. The section establishes how the manifold projection process creates the multi-dimensional cryptographic space that enables the proof-of-knowledge protocol to function. By combining Möbius rotation operations with systematic foliation algorithms, the system creates a rich, unpredictable cryptographic environment where each round produces a unique hyperplane configuration that challenges the authenticating agent's knowledge.

The importance of this section lies in its mathematical rigor and geometric intuition, which provide the foundation for understanding how the abstract

entropy values become concrete cryptographic challenges. The manifold projection represents the critical transformation phase where mathematical theory becomes implementable cryptography, creating the spatial framework that enables witness generation and verification. The geometric approach provides intuitive understanding of the system's operation while enabling sophisticated security analysis through mathematical modeling.

The interoperation with the manifold cypher and proof system is structural: the manifold projection creates the exact cryptographic space that the authenticating agent must navigate to demonstrate knowledge of the secret. The rotation operations ensure that each round presents a unique challenge, while the foliation process creates the zone structure that enables systematic witness selection. The resulting hyperplanes become the mathematical foundation for all subsequent cryptographic operations, from witness generation to final verification.

The sub-sections systematically develop the mathematical framework for manifold construction. Section 4.A introduces the Möbius rotation operation that creates circular shifts of alphabet rings, establishing the mathematical foundation for introducing randomness into the system. Section 4.B presents the n-way contiguous foliation algorithm that divides rotated rings into balanced, non-overlapping slices that form the hyperplanes. Section 4.C establishes the mathematical relationships between rotation indices and slice origins, creating the audit trail necessary for verification and replay. Together, these operations create the complete mathematical framework for transforming entropy into verifiable cryptographic structures.

4.A) Möbius Rotation

Möbius rotation by k on string s of length $|s|$:

$$\rho_k(s) = s[k : |s| - 1]circs[0 : k - 1], \quad 0 \leq k < |s|$$

The Möbius rotation ρ_k implements a circular shift operation that preserves the string's character content while changing the starting position. This rotation is named after the Möbius strip concept, as it creates a continuous, seamless transformation where characters "wrap around" the string boundary. The operation extracts the substring from position k to the end, then appends the substring from the beginning up to position $k-1$, effectively rotating the entire string by k positions clockwise.

This rotation mechanism is crucial for the proof-of-knowledge system because it ensures that every possible starting position can be reached from any seed value k. The modulo operation $0 \leq k < |s|$ guarantees that the rotation index is always valid, preventing out-of-bounds access while maintaining the mathematical properties of the transformation. For example, rotating a 30-character alphabet by 7 positions creates a completely different slice pattern than rotating by 0 or 15 positions.

4.B) N-Way Contiguous Foliation

n-way contiguous foliation of s into n slices (Agent-tunable; default $n = 6$):

$$q = \left\lfloor \frac{|s|}{n} \right\rfloor, \qquad r = |s| \bmod n$$

$$\ell_j = q + \mathbf{1}\{j < r\}, \quad a_0 = 0, \quad a_{j+1} = a_j + \ell_j$$

$$s_j = s[a_j : a_{j+1} - 1], \quad j = 0, \ldots, n-1$$

The foliation process divides a rotated string into exactly n contiguous slices, where n is configurable during the commitment phase (default $n = 6$ for this system). The algorithm first calculates the base slice length q using integer division, which represents the minimum number of characters each slice will receive. The remainder r indicates how many slices will get one extra character to handle cases where the string length isn't perfectly divisible by n.

The slice length calculation $\ell_j = q + \mathbf{1}\{j < r\}$ uses an indicator function that adds 1 to the base length for the first r slices. This ensures that all characters are distributed across the slices while maintaining as much balance as possible. The starting positions a_j are computed incrementally, with each slice beginning immediately after the previous one ends. This contiguous approach guarantees that no characters are lost or duplicated during the slicing process.

The final slice extraction $s_j = s[a_j : a_{j+1} - 1]$ creates each slice by taking characters from the calculated range. The range notation $[a_j : a_{j+1} - 1]$ ensures that slice boundaries are properly defined and that adjacent slices share no characters, maintaining the disjoint property essential for the proof-of-knowledge verification.

4.C) Recorded Per-Slice Rotation Index Progression

Recorded per-slice rotation index progression (for logging):

$$\kappa(s, n, k, j) = (k + j\, q) \bmod |s|$$

The index progression function κ tracks the mathematical relationship between the original rotation seed k and the starting position of each slice j in the foliated string. This function calculates where in the original (unrotated) string each slice originated from, providing a deterministic mapping that can be used for verification, logging, and replay purposes.

The formula $(k + j \cdot q) \bmod |s|$ represents a linear progression through the rotated string, where each slice j starts q positions further than the previous slice. The modulo operation ensures that the index wraps around the string boundary, maintaining consistency with the Möbius rotation concept. This progression is crucial for maintaining the mathematical integrity of the system, as it allows auditors to verify that the slices were generated correctly and that no manipulation occurred during the selection process.

For the default mode with $n = 6$, this function provides a systematic way to track how the 6 slices relate to each other and to the original rotation seed. This transparency is essential for the proof-of-knowledge system, as it demonstrates that the witness generation process is deterministic and verifiable rather than random or arbitrary.

For this program $n = 6$ (configurable during commitment phase). Example default sizes yield q values such as: if $|S_U| = |S_L| = 30$ then $q = 5, r = 0$; if $|S_N| = 12$ then $q = 2, r = 0$. In general, q and r depend on the Agent-committed ring sizes and n.

```
graph LR
  subgraph "Rotation"
    A["Alphabet α"] --> K["Offset k = "] --> R["Rotated α' = _k(α)"]
  end
  subgraph "Foliation _n"
    R --> Q["q = |α|/n, r = |α| mod n"]
    Q --> L["_j = q + 1{j < r}"]
    L --> S["Slices α'_0 .. α'_{n-1}"]
  end
```

5) Per-Round Hyperplane Construction

Chapter Overview

This section describes the dynamic, round-by-round construction of the cryptographic hyperplanes that form the interactive challenge space for the proof-of-knowledge protocol. The section establishes how each round generates a completely new hyperplane configuration through independent entropy-driven seeding, ensuring that knowledge of previous rounds cannot be used to predict future challenges. By implementing independent random number generation for each alphabet and round, the system creates a constantly evolving cryptographic environment that maintains unpredictability while preserving mathematical consistency and verifiability.

The importance of this section lies in its demonstration of how the theoretical manifold projection concepts become practical, implementable cryptographic operations that can be executed in real-time. The per-round construction ensures that each interaction produces a unique challenge, preventing replay attacks and ensuring that the proof-of-knowledge protocol maintains its security properties across multiple rounds. The independent seeding strategy creates the mathematical foundation for the system's resistance to correlation attacks and ensures that each round's hyperplane is statistically independent from all others.

The interoperation with the manifold cypher and proof system is dynamic: the hyperplane construction process creates the exact cryptographic challenges that the authenticating agent must solve in each round. The witnesses generated through this process become the mathematical evidence that the verification circuit uses to evaluate the proof of knowledge. The round-by-round construction

ensures that the system can scale to arbitrary numbers of rounds while maintaining security properties, enabling flexible deployment scenarios that can adapt to different security requirements.

The sub-sections provide a complete description of the hyperplane construction process. Section 5.A establishes the seed generation mechanism that provides independent randomness for each round and alphabet, ensuring unpredictability and security. Section 5.B describes the rotation operations that apply the entropy-derived offsets to create unique alphabet configurations for each round. Section 5.C explains the n-slice foliation process that divides the rotated alphabets into the zone structure that enables witness selection. Section 5.D presents the witness construction process that combines alphabet slices from different zones to create composite witnesses that span multiple character modalities. Together, these operations create the complete round-by-round hyperplane construction that enables the interactive proof-of-knowledge protocol.

5.A) Seed Generation

For round t draw seeds $k_U^t \in \{0, \ldots, |S_U| - 1\}$, $k_L^t \in \{0, \ldots, |S_L| - 1\}$, $k_N^t \in \{0, \ldots, |S_N| - 1\}$ and define:

Note: The system supports an arbitrary number of hyperplanes, with the specific configuration determined during the commitment phase by the authenticating Agent. The default implementation uses 6 zones, but this is configurable based on the Agent's specifications.

5.B) Rotations

Rotations:

$$\hat{U}^t = \rho_{k_U^t}(S_U), \qquad \hat{L}^t = \rho_{k_L^t}(S_L), \qquad \hat{N}^t = \rho_{k_N^t}(S_N)$$

The rotation phase generates three independently rotated versions of the base alphabets using randomly selected seeds for each round. Each seed k_U^t, k_L^t, and k_N^t is drawn from the uniform distribution over the respective alphabet's index range, ensuring that each round produces a completely different character arrangement. The hat notation ($\hat{U}^t$, $\hat{L}^t$, $\hat{N}^t$) distinguishes these rotated alphabets from the original base alphabets, emphasizing their transformed state.

This independent seeding strategy is crucial for the security of the proof-of-knowledge system, as it prevents correlation between rounds and ensures that knowledge of previous rounds' witness patterns cannot be used to predict future rounds. The random selection of seeds makes each round's hyperplane construction unpredictable while maintaining the deterministic mathematical relationships within each round. This randomness is essential for preventing replay attacks and ensuring that each proof attempt is unique.

5.C) N-Slice Foliations

N-slice foliations (default $n = 6$): $(U_0^t, \ldots, U_{n-1}^t) = \Pi_n(\hat{U}^t), \quad (L_0^t, \ldots, L_{n-1}^t) = \Pi_n(\hat{L}^t), \quad (N_0^t, \ldots, N_{n-1}^t) = \Pi_n(\hat{N}^t)$

The foliation phase applies the 6-way slicing algorithm to each rotated alphabet, creating six distinct slices per alphabet per round. This process transforms the continuous rotated strings into discrete, manageable chunks that can be individually selected and combined. Each slice U_j^t, L_j^t, and N_j^t represents a specific zone j within the round t hyperplane, providing the building blocks for witness construction.

The parallel foliation of all three alphabets ensures that corresponding zones across alphabets are aligned and can be meaningfully combined. This alignment is essential for maintaining the mathematical consistency of the system, as it guarantees that zone j in the uppercase alphabet corresponds to zone j in the lowercase and numeric alphabets. The resulting slice structure creates a 6×3 grid of character sets, where each cell contains characters from a specific alphabet and zone combination.

5.D) Zone-J Witness (Emergent Hyperplane Slice)

Zone-j witness (emergent hyperplane slice):

$$W_j^t = U_j^t \circ L_j^t \circ N_j^t$$

The witness construction combines the three alphabet slices for a given zone into a single composite string that represents the full character space available in that zone for that round. This concatenation creates a rich, multi-dimensional witness that contains characters from all three character classes, significantly increasing the probability that any given secret character will be found within at least one witness.

The term "emergent hyperplane slice" emphasizes that the witness is not simply a predefined set of characters, but rather emerges from the mathematical combination of independently generated slices. This emergence property ensures that the witness content is unpredictable and unique to each round, while maintaining the deterministic mathematical relationships that make the system verifiable. The witness serves as the fundamental unit of proof in the system, representing the authenticating agent's knowledge of which zone to select in each round.

Lengths in default mode: $|U_j^t| = 5,\ |L_j^t| = 5,\ |N_j^t| = 2 \Rightarrow |W_j^t| = 12.$

```
sequenceDiagram
  autonumber
  participant G as "RNG"
  participant U as "S_U"
  participant L as "S_L"
  participant N as "S_N"
  participant F as "Foliator _n"
  participant W as "Witness"
  G->>U: "k_U^t"
  G->>L: "k_L^t"
  G->>N: "k_N^t"
```

```
U->>F: "_{k_U^t}(S_U)"
L->>F: "_{k_L^t}(S_L)"
N->>F: "_{k_N^t}(S_N)"
F-->>W: "W_j^t = U_j^t  L_j^t  N_j^t"
```

6) Witness Morphism: Color/Zone ↔ Input Symbols

Chapter Overview

This section establishes the critical bridge between human cognitive processes and mathematical cryptographic verification through the witness morphism system that maps user interface inputs to mathematical zone indices. The section introduces the concept of private morphisms that create a security-through-obscurity layer while enabling intuitive human-computer interaction. By establishing bijective mappings between input symbols and hyperplane zones, the system creates a projective interface that allows authenticating agents to navigate the cryptographic space naturally while maintaining the mathematical rigor necessary for security applications.

The importance of this section lies in its fundamental innovation of creating a human-centric approach to interactive cryptography that maintains strong security properties. The morphism system represents a departure from traditional cryptographic protocols that require precise mathematical operations, instead allowing humans to interact naturally through familiar interface elements while preserving cryptographic security. The private nature of the morphism adds an additional layer of security through obscurity, as adversaries cannot determine the input mapping without additional information about the system's internal logic.

The interoperation with the manifold cypher and proof system is interactive: the morphism system translates human input actions into the mathematical zone selections that drive the witness generation and verification processes. The zone selections determine which witnesses are captured for each round, and these witnesses become the mathematical evidence that the verification circuit uses to evaluate the proof of knowledge. The morphism ensures that the human-computer interaction layer is completely separated from the mathematical verification layer, maintaining clean security separation while enabling intuitive operation.

Within we develop the morphism concept from basic principles to practical implementation. Section 6.A introduces the legend glyphs that represent the input symbols available to authenticating agents, establishing the visual language of the projective interface. Section 6.B presents example default mappings that demonstrate how input symbols can be mapped to zone indices, showing both standard and reversed configurations that provide flexibility in interface design. Together, these sections establish the complete framework for translating human input into mathematical cryptographic operations while maintaining security and usability.

6.A) Legend Glyphs

Legend glyphs: , , , , /, \.

Important: The specific mapping between input symbols and zone indices is established by the authenticating Agent during the commitment phase. The mappings shown below are examples of how such a system might be configured, but the actual mapping is determined by the Agent's specifications.

6.B) Example Default Mapping

Example default mapping (`DIRECTION_SWITCH` = false):

$$m(\ ) = 1, \quad m(\ ) = 4, \quad m(\ ) = 2, \quad m(\ ) = 0, \quad m(/) = 5, \quad m(\backslash) = 3$$

This example mapping function m establishes a direct correspondence between authenticating agent input symbols and zone indices, creating an intuitive projective interface where directional keys map to spatially related zones. The arrow keys (, , ,) are mapped to zones that form a logical spatial pattern, with the up arrow () corresponding to zone 1 (YELLOW), down arrow () to zone 4 (GREEN), right arrow () to zone 2 (RED), and left arrow () to zone 0 (BLUE). The slash symbols (/,) are assigned to zones 5 (BLACK) and 3 (WHITE) respectively, providing additional input options.

This mapping design considers human cognitive patterns and ergonomics, placing frequently used directional inputs in easily accessible zones. The spatial relationship between arrow directions and zone assignments creates a mental model that authenticating agents can quickly internalize, reducing cognitive load during the proof-of-knowledge process. The mapping also ensures that all six zones are accessible through standard keyboard inputs, making the system usable across different input devices and authenticating agent preferences.

- Example reversed mapping (`DIRECTION_SWITCH` = true):

  $$m_r(\ ) = 2, \quad m_r(\ ) = 1, \quad m_r(\ ) = 3, \quad m_r(\ ) = 0, \quad m_r(/) = 5, \quad m_r(\backslash) = 4$$

  This example reversed mapping function m_r provides an alternative key-to-zone assignment that maintains the same mathematical structure while offering different authenticating agent projective interface layouts. This reversal primarily affects the arrow key mappings, with the up arrow () now corresponding to zone 2 (RED) instead of zone 1, and the down arrow () mapping to zone 1 (YELLOW) instead of zone 4. The right arrow () maps to zone 3 (WHITE), and the left arrow () remains at zone 0 (BLUE) for consistency. The availability of two distinct mappings serves several important purposes in the proof-of-knowledge system. First, it provides flexibility for different authenticating agent projective interface designs and preferences, allowing the same mathematical core to support multiple interaction paradigms. Second, it adds an additional layer of configuration that can be used to customize the system for different applications or

authenticating agent groups. Third, it demonstrates the robustness of the underlying algorithm, showing that the proof-of-knowledge mechanics are independent of the specific input mapping used.

These realize a morphism from the 6 color zones to 6 distinct input symbols via the zone index.

```
graph LR
  K["Input key (    / \\)"] --> DS{"DIRECTION_SWITCH?"}
  DS -- "false" --> M["m: key → zone index"]
  DS -- "true" --> MR["m_r: key → zone index"]
  M --> Z["Selected zone z_t"]
  MR --> Z
```

7) Interactive Selection and Collection

Chapter Overview

This section establishes the core interactive mechanics of the Rosario-Wang proof-of-knowledge protocol, describing how authenticating agents navigate the cryptographic manifold through human-computer interaction to generate verifiable witnesses. The section presents the mathematical framework that translates human input actions into mathematical zone selections, witness captures, and evidence accumulation, creating the bridge between intuitive human operation and rigorous cryptographic verification. By implementing a systematic collection and validation process, the system ensures that each round produces verifiable evidence that can be independently reconstructed and audited.

The importance of this section lies in its demonstration of how the theoretical manifold projection concepts become practical, interactive cryptographic operations that maintain security while enabling human-centric operation. The interactive selection process represents the critical interface between human cognitive processes and mathematical cryptographic verification, where the authenticating agent's knowledge of the secret must be demonstrated through systematic navigation of the hyperplane space. The collection mechanisms ensure that all evidence is properly captured, validated, and accumulated for final verification.

The interoperation with the manifold cypher and proof system is operational: the interactive selection process determines which witnesses are captured in each round, and these witnesses become the mathematical evidence that drives the final verification process. The zone selection function translates human input into mathematical indices, the witness capture process extracts the relevant character content, and the collection accumulation creates the complete audit trail necessary for verification. The system maintains mathematical consistency while providing intuitive human operation.

The sub-sections provide a comprehensive description of the interactive selection and collection process. Section 7.A establishes the zone selection function that maps authenticating agent inputs to mathematical zone indices, ensuring deterministic and verifiable selection. Section 7.B describes the witness and

index capture process that extracts the character content and mathematical relationships for each round. Section 7.C presents the collection accumulation mechanisms that build the complete evidence set across all rounds. Section 7.D covers input processing and validation to ensure data integrity. Section 7.E explains round synchronization and state management for maintaining mathematical consistency. Section 7.F addresses error handling and recovery mechanisms to ensure robust operation. Together, these components create the complete interactive framework for proof-of-knowledge generation.

7.A) Zone Selection Function

Given authenticating agent keys $(k_1, \ldots, k_T)$, define zone choices:

$$z_t = \begin{cases} m(k_t), & \text{if DIRECTION\_SWITCH=false} \\ \\ m_r(k_t), & \text{if DIRECTION\_SWITCH=true} \end{cases}$$

The zone selection function z_t determines which zone the authenticating agent has selected in each round based on their input key k_t and the current direction setting. This function acts as a bridge between the authenticating agent projective interface layer and the mathematical core of the proof-of-knowledge system, translating human input actions into mathematical zone indices that can be processed by the algorithm. The conditional structure ensures that the appropriate mapping function (m or m_r) is applied based on the system configuration.

The zone selection process is deterministic and verifiable, as each input key k_t maps to exactly one zone index z_t regardless of when the input occurs. This determinism is essential for the proof-of-knowledge system, as it ensures that the same sequence of authenticating agent inputs will always produce the same sequence of zone selections. The mapping functions m and m_r are designed to be bijective, meaning that each zone index is reachable through exactly one input symbol, preventing ambiguity in the selection process.

7.B) Witness and Index Capture

Witness and index captured at round t:

$$X_t = W_{z_t}^t, \qquad I_t = (\kappa(S_U, n, k_U^t, z_t),\ \kappa(S_L, n, k_L^t, z_t),\ \kappa(S_N, n, k_N^t, z_t))$$

For each round t, the system captures two critical pieces of information: the witness string X_t and the index triple I_t. The witness X_t represents the actual character content that the authenticating agent has selected by choosing zone z_t, serving as the primary evidence in the proof-of-knowledge verification. This witness contains characters from all three alphabets (uppercase, lowercase, and numeric) that were available in the selected zone during that specific round.

The index triple I_t provides a mathematical audit trail that records exactly how the witness was generated. Each component of the triple represents the rotation index that produced the corresponding alphabet slice in the witness. This index information is crucial for verification purposes, as it allows auditors to reconstruct the exact mathematical process that generated each witness. The index triple also serves as a deterministic record that can be used to replay the witness generation process, ensuring that the system's behavior is reproducible and verifiable.

7.C) Collection Accumulation

Collections: $X^{1:T} = (X_1, \ldots, X_T)$, $Z^{1:T} = (z_1, \ldots, z_T)$, $I^{1:T} = (I_1, \ldots, I_T)$. The collection notation $X^{1:T}$, $Z^{1:T}$, and $I^{1:T}$ represents the accumulated evidence from all rounds of the proof-of-knowledge interaction. These collections serve as the complete record of the authenticating agent's interaction with the system, providing all the information necessary for verification and analysis. The superscript notation $1:T$ indicates that the collection spans from round 1 to round T, where T is the total number of rounds completed.

The witness collection $X^{1:T}$ contains all the character strings that the authenticating agent has selected across all rounds, forming the primary evidence for the proof-of-knowledge claim. The zone collection $Z^{1:T}$ records the sequence of zone selections, providing insight into the authenticating agent's decision-making pattern and the mathematical structure of their selections. The index collection $I^{1:T}$ maintains the complete audit trail, enabling full reconstruction and verification of the witness generation process. Together, these three collections provide a comprehensive view of the proof-of-knowledge interaction that can be analyzed, verified, and audited.

7.D) Input Processing and Validation

The system processes authenticating agent inputs through a multi-stage validation pipeline that ensures data integrity and consistency. Each input key k_t undergoes validation to ensure it corresponds to a valid zone mapping, and the resulting zone selection z_t is verified to be within the valid range $[0, n-1]$. This validation prevents invalid inputs from corrupting the proof-of-knowledge process and ensures that all zone selections are mathematically sound.

7.E) Round Synchronization and State Management

Each round maintains synchronized state across all system components, ensuring that the manifold construction, zone selection, and witness capture occur in the correct sequence. The round counter t serves as a synchronization mechanism that coordinates the entropy-driven offset generation, alphabet rotation, and hyperplane foliation. This synchronization ensures that each round produces a consistent and verifiable mathematical structure that can be independently reconstructed using the logged indices and entropy values.

7.F) Error Handling and Recovery Mechanisms

The system implements robust error handling for various failure scenarios, including invalid inputs, network interruptions, and computational errors. When errors occur, the system can recover gracefully by maintaining the current round state and allowing the authenticating agent to retry the input. This resilience ensures that the proof-of-knowledge process can continue even in the presence of transient failures, while maintaining the mathematical integrity of the verification process.

```
sequenceDiagram
  autonumber
  participant A as "Agent"
  participant UI as "Projection UI"
  participant P as "Morphism (m/m_r)"
  participant M as "Manifold"
  participant W as "Witness Log"
  loop "For t = 1..T"
    A->>UI: "Press key k_t"
    UI->>P: "Map k_t"
    P-->>UI: "z_t"
    UI->>M: "Get W_{z_t}^t, I_t"
    M-->>UI: "X_t, I_t"
    UI->>W: "Append (X_t, I_t)"
  end
```

8) Verifier Rule (Proof-of-Knowledge)

Chapter Overview

This section presents the mathematical foundation of the Rosario-Wang proof-of-knowledge verification system, establishing the formal rules and algorithms that determine whether an authenticating agent has successfully demonstrated knowledge of the secret. The section defines the acceptance conditions, verification algorithms, and decision mechanisms that transform the accumulated witness evidence into a definitive cryptographic proof. By implementing rigorous mathematical verification with clear acceptance criteria, the system provides deterministic and auditable proof-of-knowledge validation that maintains security while enabling practical deployment.

The importance of this section lies in its establishment of the mathematical rigor that makes the proof-of-knowledge protocol cryptographically sound and practically verifiable. The verifier rule represents the culmination of the entire cryptographic process, where all the mathematical relationships, entropy-driven offsets, and witness collections are evaluated against the secret to determine success or failure. The acceptance conditions ensure that partial knowledge is insufficient, requiring complete demonstration of the secret across all required rounds while maintaining mathematical consistency and verifiability.

The interoperation with the manifold cypher and proof system is evaluative: the verifier rule processes the complete evidence set generated through the interactive selection and collection process, applying mathematical verification algorithms to determine whether the proof-of-knowledge claim is valid. The verification process uses the logged indices and entropy values to independently reconstruct the mathematical relationships that generated each witness, ensuring that the verification is deterministic and reproducible. The system maintains complete auditability while providing clear, actionable verification results.

The sub-sections systematically develop the verification framework from basic principles to practical implementation. Section 8.A establishes the secret structure and requirements that define what constitutes valid knowledge in the system. Section 8.B presents the formal acceptance condition that mathematically defines when verification succeeds. Section 8.C introduces the acceptance indicator as an alternative mathematical formulation that enables efficient computation. Section 8.D establishes the final decision rule that translates mathematical results into clear outcomes. Section 8.E describes the verification algorithm implementation that executes the mathematical verification process. Section 8.F addresses performance optimization and scalability considerations for practical deployment. Section 8.G covers verification result persistence and audit capabilities for compliance and analysis. Together, these components create the complete verification framework that ensures cryptographic security and mathematical integrity.

8.A) Secret Structure and Requirements

Let the secret be $s = s_1 s_2 \cdots s_L$.
The secret string s represents the knowledge that the authenticating agent is attempting to prove, with each character s_i representing a component of that knowledge. The length L of the secret determines the minimum number of rounds required for a successful proof, as the system needs at least L witnesses to potentially contain all the secret characters. Verification is case-sensitive and applies no normalization; this increases the effective combinatorial space and resists brute force.

The secret's structure as a sequence of characters $s_1 s_2 \cdots s_L$ allows the proof-of-knowledge system to verify knowledge on a character-by-character basis. This granular approach ensures that partial knowledge is not sufficient for acceptance, requiring the authenticating agent to demonstrate complete knowledge of the entire secret. The sequential indexing also enables the system to establish a one-to-one correspondence between secret characters and witness rounds, creating a systematic verification framework.

8.B) Acceptance Condition

Acceptance condition:

$$\text{ACCEPT} \iff T \geq L \ \wedge \ \bigwedge_{i=1}^{L} \mathbf{1}\{\, s_i \in \text{chars}(X_i) \,\} = 1$$

The acceptance condition consists of two fundamental requirements that must both be satisfied for the proof-of-knowledge to succeed. The first requirement $T \geq L$ ensures that the authenticating agent has provided enough rounds to potentially contain all the secret characters. This is a necessary but not sufficient condition, as having enough rounds doesn't guarantee that the right characters were selected in each round.

The second requirement uses the logical AND operator ($\bigwedge$) to ensure that every character s_i in the secret is found (case-sensitive, no normalization) within the corresponding witness X_i. The indicator function $\mathbf{1}\{\cdot\}$ returns 1 if the condition is true and 0 otherwise, making the entire expression evaluate to 1 only when all characters are successfully found. The character extraction function chars$(\cdot)$ focuses the search on the actual character content rather than formatting or structure.

8.C) Acceptance Indicator

Equivalently define the acceptance indicator:

$$A(s, X^{1:T}) = \mathbf{1}\{T \geq L\} \ \prod_{i=1}^{L} \mathbf{1}\{\, s_i \in X_i \,\}$$

The acceptance indicator $A(s, X^{1:T})$ provides an alternative mathematical formulation of the verification rule that is mathematically equivalent but computationally more convenient. The indicator function $\mathbf{1}\{T \geq L\}$ ensures the round count requirement is met, while the product $\prod_{i=1}^{L}$ multiplies together all the individual character verification results. Since each indicator function returns either 0 or 1, the product will equal 1 only when all individual verifications succeed, and 0 if any single verification fails.

This formulation has several advantages: it provides a single numerical value that represents the overall verification result, it makes it easy to identify which specific characters failed verification (by examining individual factors), and it enables efficient computation through simple arithmetic operations. The acceptance indicator also makes it clear that the verification process is multiplicative rather than additive, emphasizing that all conditions must be satisfied simultaneously rather than being satisfied to some degree.

8.D) Final Decision Rule

Return `PASS` iff $A = 1$, else `FAIL`.
The final decision rule translates the mathematical acceptance indicator into a clear, actionable result. When $A = 1$, all verification conditions have been met, and the system returns `PASS`, indicating that the authenticating agent has

successfully demonstrated knowledge of the secret. When $A = 0$, at least one verification condition has failed, and the system returns `FAIL`, indicating that the proof-of-knowledge attempt was unsuccessful.

This binary outcome provides clear feedback to both the authenticating agent and any external systems that might be using the proof-of-knowledge verification. The `PASS`/`FAIL` terminology is intuitive and unambiguous, making it easy for authenticating agents to understand their verification status. The deterministic nature of this decision rule ensures that the same input will always produce the same result, maintaining the mathematical integrity and verifiability of the system.

8.E) Verification Algorithm Implementation

The verification algorithm implements the acceptance condition through a systematic character-by-character search across all witnesses. For each secret character s_i, the algorithm searches the corresponding witness X_i using case-sensitive string matching. The algorithm maintains a verification state that tracks which characters have been successfully found and which remain to be verified, enabling early termination when verification becomes impossible.

8.F) Performance Optimization and Scalability

The verification process is optimized for performance through efficient string matching algorithms and parallel processing capabilities. For large secrets or high-volume verification scenarios, the system can process multiple verification requests concurrently while maintaining mathematical correctness. The verification algorithm scales linearly with the secret length L and can handle secrets of arbitrary length limited only by system resources and commitment parameters.

8.G) Verification Result Persistence and Audit

All verification results are persistently stored with complete audit trails, including the input witnesses, verification decisions, and computational timestamps. This persistence enables historical analysis, performance monitoring, and compliance auditing. The audit trail includes both successful and failed verification attempts, providing comprehensive visibility into the system's operation and security posture.

```
graph TD
  S(["Start"]) --> C1{"T ≥ L ?"}
  C1 -- "No" --> F1["FAIL"]
  C1 -- "Yes" --> C2["Check s_i  X_i for i=1..L"]
  C2 --> C3{"All true ?"}
  C3 -- "No" --> F2["FAIL"]
  C3 -- "Yes" --> P["PASS"]
```

9) Alphabet Distribution and Properties (Configurable Mode)

Chapter Overview

This section examines the mathematical properties and distribution characteristics of the alphabet systems that form the foundation of the Rosario-Wang cryptographic architecture, focusing on how character sets are systematically distributed across hyperplanes to create balanced and secure cryptographic challenges. The section analyzes the geometric and statistical properties of alphabet slicing, the mathematical relationships between different character modalities, and the optimization strategies that ensure uniform distribution across all zones. By establishing the mathematical framework for alphabet distribution, the system creates a robust foundation for witness generation that maintains security while enabling flexible configuration and deployment.

The importance of this section lies in its demonstration of how mathematical principles of distribution theory and combinatorics are applied to create cryptographically sound alphabet systems. The alphabet distribution represents the fundamental building block of the proof-of-knowledge protocol, as the quality and balance of character distribution directly impacts the security and usability of the system. The mathematical analysis of distribution properties ensures that no zone is systematically advantaged or disadvantaged, maintaining the fairness and unpredictability necessary for cryptographic security while enabling efficient witness generation and verification.

The interoperation with the manifold cypher and proof system is foundational: the alphabet distribution creates the mathematical structure that determines how entropy-driven offsets are applied to create unique hyperplane configurations in each round. The distribution properties ensure that the rotation and foliation operations produce mathematically consistent and verifiable results, while the balance characteristics guarantee that all zones provide equivalent cryptographic challenges. The configurable nature of the distribution system enables adaptation to different security requirements and deployment scenarios.

The sub-sections provide a comprehensive analysis of alphabet distribution from general principles to specific implementations. Section 9.A establishes the general hyperplane distribution framework that applies to arbitrary alphabet sizes and zone counts, providing the mathematical foundation for all distribution operations. Section 9.B analyzes the uppercase alphabet distribution, examining the specific mathematical properties of the 30-character set in the default 6-zone configuration. Section 9.C examines the lowercase alphabet distribution, demonstrating how independent seeding maintains security while preserving mathematical consistency. Section 9.D investigates the numeric alphabet distribution, showing how smaller character sets are optimally distributed across zones. Section 9.E establishes witness length consistency properties that ensure uniform cryptographic challenges. Section 9.F presents alphabet balancing and optimization strategies for maintaining security across different alphabet configurations. Section 9.G describes dynamic alphabet configuration capabilities that enable adaptive security deployment. Section 9.H provides cross-alphabet correlation analysis to ensure mathematical independence and

security. Together, these analyses create the complete mathematical framework for alphabet distribution that underpins the entire cryptographic system.

9.A) General Hyperplane Distribution

General n hyperplanes: for any ring of length $|s|$ and hyperplane count $n \geq 2$ chosen by the Agent, lengths follow

$$q = \left\lfloor \frac{|s|}{n} \right\rfloor, \quad r = |s| \bmod n, \quad \ell_j = q + \mathbf{1}\{j < r\},\ j = 0, \ldots, n-1,$$

yielding disjoint, contiguous coverage. The bullets below describe the default $n = 6$ instance.

9.B) Upper Alphabet Distribution

- Upper per round: each zone receives exactly 5 contiguous characters (disjoint, covering all 30) in the default 6-zone configuration.
  The uppercase alphabet distribution ensures that each of the six zones receives exactly 5 characters per round, with the total of 30 characters being perfectly divisible by 6. This uniform distribution creates a balanced character allocation where no zone is advantaged or disadvantaged in terms of character count. The contiguous nature of the slices means that characters within each zone are adjacent in the original alphabet, maintaining some semantic relationships while ensuring randomness through the rotation process.

  The disjoint property guarantees that no character appears in multiple zones within the same round, preventing overlap and ensuring that each character is uniquely assigned to exactly one zone. This property is crucial for the mathematical integrity of the system, as it prevents double-counting of characters and ensures that the total character coverage across all zones equals the full alphabet size. The complete coverage property means that every character in the uppercase alphabet appears in exactly one zone per round, leaving no characters unassigned.

9.C) Lower Alphabet Distribution

- Lower per round: same as upper, independently seeded.
  The lowercase alphabet follows the identical distribution pattern as the uppercase alphabet, with each zone receiving exactly 5 characters per round. However, the independent seeding means that the rotation patterns for uppercase and lowercase alphabets are completely uncorrelated, even though they follow the same mathematical structure. This independence is essential for the security of the proof-of-knowledge system, as it prevents attackers from using knowledge of one alphabet's distribution to predict the other's.

The parallel structure between uppercase and lowercase alphabets ensures mathematical consistency and predictability in the system design, while the independent seeding maintains the randomness and unpredictability necessary for security. This design choice demonstrates the balance between mathematical elegance and cryptographic security, where the structure provides verifiability while the independence provides unpredictability.

9.D) Numeric Alphabet Distribution

- Numeric per round: each zone receives exactly 2 contiguous characters (disjoint, covering all 12) in the default 6-zone configuration.
  The numeric alphabet distribution follows the same mathematical principles as the alphabetic alphabets but with a smaller slice size due to the smaller alphabet size. With only 12 characters, each zone receives exactly 2 characters per round, maintaining the uniform distribution principle. The smaller slice size means that each numeric zone contains fewer characters, but this is compensated by the fact that numeric characters are often more distinctive and easier to identify within a witness.

  The numeric alphabet's perfect divisibility by 6 ($12 \div 6 = 2$) ensures that the distribution is completely uniform with no remainder to distribute. This mathematical simplicity makes the numeric component more predictable and easier to verify, while still contributing to the overall complexity of the witness strings. The numeric characters provide diversity in the character space, making the witnesses more information-rich and increasing the probability that any given secret character will be found.

9.E) Witness Length Consistency

- Thus in the default example $|W_j^t| = 12$ for all zones and rounds. In general, witness length is $|W_j^t| = \sum_a \ell_{a,j}$ over all committed rings, which depends on the Agent's alphabet sizes and chosen n.

  The consistent length property has several important implications: it ensures that all witnesses have equal information content, preventing any zone from being advantaged by having more characters to work with; it simplifies the verification algorithm by eliminating the need to handle variable-length witnesses; and it provides a predictable mathematical structure that can be easily analyzed and verified. This uniformity is essential for maintaining the mathematical integrity of the system while ensuring that the proof-of-knowledge process is fair and unbiased across all zones.

9.F) Alphabet Balancing and Optimization

The system provides mechanisms for balancing alphabets of different sizes to ensure optimal distribution across hyperplanes. When alphabets have different

cardinalities, the system can apply padding or balancing techniques to maintain uniform slice distributions. This balancing ensures that no zone is systematically disadvantaged by receiving consistently smaller character sets, maintaining the fairness and security properties of the proof-of-knowledge system.

9.G) Dynamic Alphabet Configuration

The system supports dynamic reconfiguration of alphabets during operation, allowing Agents to adjust alphabet contents, sizes, or hyperplane counts based on changing security requirements or performance constraints. This dynamic configuration capability enables the system to adapt to evolving threat models while maintaining the mathematical integrity and security properties established during the commitment phase.

9.H) Cross-Alphabet Correlation Analysis

The system provides tools for analyzing correlations between different alphabets to ensure that they provide truly independent character spaces. This analysis helps identify potential weaknesses in alphabet selection and enables optimization of alphabet combinations for maximum security. The correlation analysis considers both statistical properties and semantic relationships between characters across different alphabets.

```
graph TD
  CM["Commitment: M, {α}, n"]:::core

  subgraph A["9.A General Hyperplane Distribution"]
    A1["q = |s|/n"]
    A2["r = |s| mod n"]
    A3["_j = q + 1{j < r}"]
    A4["s → slices s_0..s_{n-1} (disjoint, contiguous, cover) [E2]"]
    A1 --> A3
    A2 --> A3
    A3 --> A4
  end

  CM --> A1
  CM --> A2

  classDef core fill:#f3f7ff,stroke:#4169e1,color:#111;

graph TD
  CM["Commitment: {S_U,S_L,S_N}, n=6 (default)"]:::core

  subgraph B["9.B Upper"]
    B1["|S_U|=30"] --> B2["q_U=5, r=0"]
    B2 --> B3["_{U,j}=5"]
```

```
    B3 --> B4["U_j^t"]
  end

  subgraph C["9.C Lower"]
    C1["|S_L|=30"] --> C2["q_L=5, r=0"]
    C2 --> C3["_{L,j}=5"]
    C3 --> C4["L_j^t"]
  end

  subgraph D["9.D Numeric"]
    D1["|S_N|=12"] --> D2["q_N=2, r=0"]
    D2 --> D3["_{N,j}=2"]
    D3 --> D4["N_j^t"]
  end

  CM --> B1
  CM --> C1
  CM --> D1

  subgraph E["9.E Witness Length Consistency"]
    E1["|W_j^t| = _a _{a,j}  [E4]"] --> E2["Default: 5+5+2 = 12"]
  end

  B4 --> E1
  C4 --> E1
  D4 --> E1

  classDef core fill:#f3f7ff,stroke:#4169e1,color:#111;

graph TD
  subgraph F["9.F Balancing & Optimization"]
    F1["Balance/pad alphabets (e.g., add ♠♣)"]
    F2["Goal: uniform _{a,j} across zones"]
    F1 --> F2
  end

  subgraph G["9.G Dynamic Alphabet Configuration"]
    G1["Adjust M, n, |α_a|, contents (at commitment)"]
    G2["Recompute q, r, _{a,j}"]
    G1 --> G2
  end

  subgraph H["9.H Cross-Alphabet Correlation Analysis"]
    H1["Measure cross-ring correlations"]
    H2["Ensure independence / minimize leakage"]
    H1 --> H2
```

```
end

F2 --> G1
H2 --> F1
G2 --> R1["Update 9.A-9.D parameters"]
R1 --> R2["Refresh distributions and |W_j^t|"]

classDef note fill:#fff7e6,stroke:#f0a500,color:#111;
```

10) Logged Indices and Determinism (Configurable Mode)

Chapter Overview

This section establishes the mathematical framework for deterministic replay and auditability in the Rosario-Wang cryptographic system, describing how logged indices enable complete reconstruction and verification of the proof-of-knowledge process while maintaining the unpredictability necessary for security. The section presents the mathematical relationships between entropy-driven seeds, rotation offsets, and zone selections that create the audit trail, demonstrating how the system balances cryptographic randomness with mathematical determinism. By implementing comprehensive index logging and deterministic replay capabilities, the system provides unprecedented transparency and verifiability in interactive cryptography while maintaining strong security properties.

The importance of this section lies in its demonstration of how the system achieves the seemingly contradictory goals of cryptographic unpredictability and mathematical determinism. The logged indices represent the critical bridge between the entropy-driven randomness that ensures security and the mathematical relationships that enable verification and audit. This balance is essential for practical deployment, as it allows external auditors to verify system operation without compromising the security properties that make the system resistant to attacks. The deterministic replay capability provides a level of transparency that is rare in cryptographic systems while maintaining the mathematical rigor necessary for security applications.

The interoperation with the manifold cypher and proof system is analytical: the logged indices provide the mathematical foundation for reconstructing the exact hyperplane configurations that were presented to authenticating agents in each round, enabling independent verification of the entire proof-of-knowledge process. The index calculations maintain the mathematical relationships between entropy values, rotation seeds, and zone selections, creating a complete audit trail that can be used to verify system operation, debug issues, and ensure compliance with security requirements. The deterministic nature of these calculations ensures that verification is reproducible and reliable.

The sub-sections systematically develop the index logging and determinism framework from mathematical foundations to practical implementation. Section 10.A establishes the general index calculation framework that applies to arbitrary alphabet sizes and zone configurations, providing the mathematical foundation for all index operations. Section 10.B presents the default configuration

indices that demonstrate the specific mathematical relationships in the 6-zone system with 30-character alphabets. Section 10.C describes the independent seed generation process that ensures cryptographic randomness while maintaining mathematical consistency. Section 10.D explains the deterministic replay capability that enables complete verification and audit of proof-of-knowledge sessions. Section 10.E addresses index integrity and tamper detection mechanisms that protect the audit trail from manipulation. Section 10.F covers index compression and storage optimization techniques for efficient long-term retention. Section 10.G describes real-time index validation that ensures mathematical correctness during operation. Section 10.H provides cross-round index correlation analysis to identify potential weaknesses in the entropy-driven offset generation. Together, these components create the complete framework for mathematical transparency and cryptographic auditability that distinguishes the Rosario-Wang system from traditional cryptographic approaches.

10.A) General Index Calculation

For general n with ring-specific base slice lengths $q_U = \lfloor |S_U|/n \rfloor$, $q_L = \lfloor |S_L|/n \rfloor$, $q_N = \lfloor |S_N|/n \rfloor$, the logged indices for a selection of zone j in round t are

$$I_t^{(\text{general})} = ((k_U^t + q_U j) \bmod |S_U|,\ (k_L^t + q_L j) \bmod |S_L|,\ (k_N^t + q_N j) \bmod |S_N|).$$

10.B) Default Configuration Indices

In the default 6-zone configuration this simplifies to:
For a selection of zone j in round t:

$$I_t = ((k_U^t + 5j) \bmod 30,\ (k_L^t + 5j) \bmod 30,\ (k_N^t + 2j) \bmod 12)$$

The index calculation formula provides a deterministic way to compute the exact rotation positions that generated each slice in the witness for zone j in round t. The formula uses the base slice lengths (5 for uppercase and lowercase, 2 for numeric) multiplied by the zone index j, then adds the rotation seed for that round and applies the modulo operation to wrap around the alphabet boundaries. This calculation ensures that the index progression follows the same mathematical pattern used in the foliation process.

Note: The specific slice lengths (5, 5, 2) and the number of zones (6) are determined during the commitment phase by the authenticating Agent. The formula shown above represents the default configuration but can be adjusted based on the Agent's specifications.

The three components of the index triple correspond to the three alphabets: the first component $(k_U^t + 5j) \bmod 30$ represents the starting position in the uppercase alphabet, the second component $(k_L^t + 5j) \bmod 30$ represents the starting position in the lowercase alphabet, and the third component $(k_N^t + 2j) \bmod 12$ represents the starting position in the numeric alphabet. Each component is

calculated independently using the appropriate alphabet size and slice length, ensuring mathematical consistency across all three character sets.

The modulo operations ensure that all indices remain within the valid range for their respective alphabets: 0-29 for uppercase and lowercase (30 characters), and 0-11 for numeric (12 characters). This boundary handling is essential for maintaining the mathematical integrity of the system and preventing index overflow errors. The deterministic nature of these calculations means that given the same seeds and zone selection, the same indices will always be produced, enabling reliable verification and replay of the witness generation process.

Here seeds k_U^t, k_L^t, k_N^t are derived from independent calls to the shell RNG and recorded per round.

The independent random number generation for each alphabet ensures that the rotation patterns are completely uncorrelated across alphabets and rounds, providing the randomness necessary for cryptographic security. Each seed is generated independently using the system's random number generator, preventing any predictable relationships between different alphabets or between different rounds. This independence is crucial for preventing attacks that might attempt to exploit correlations in the rotation patterns.

The recording of seeds per round serves multiple important purposes: it provides an audit trail that can be used to verify the mathematical correctness of the witness generation process; it enables replay and debugging of specific rounds for analysis purposes; and it maintains the deterministic nature of the system while preserving the randomness of the seed selection. The combination of random seed generation and deterministic index calculation creates a system that is both unpredictable and verifiable, balancing security requirements with mathematical rigor.

This logging approach demonstrates the system's commitment to transparency and verifiability, as every mathematical operation can be reconstructed and verified using the recorded seeds and the deterministic formulas. This transparency is essential for the proof-of-knowledge system, as it allows external auditors to verify that the system is operating correctly and that no manipulation has occurred during the witness generation or verification processes.

10.C) Independent Seed Generation

The system generates independent random seeds for each alphabet and round to ensure maximum unpredictability and security. Each seed k_α^t is generated using cryptographically secure random number generation techniques, ensuring that the rotation patterns cannot be predicted or reproduced by any computational means. The independence between seeds prevents correlation attacks and ensures that knowledge of one alphabet's rotation pattern provides no information about other alphabets or rounds.

10.D) Deterministic Replay Capability

The logged indices enable deterministic replay of the entire proof-of-knowledge session, allowing auditors to independently verify the mathematical correctness of the witness generation process. Given the original entropy $\mathcal{E}$, time coefficient τ, and the logged seeds, any party can reconstruct the exact manifold that was presented to the authenticating agent. This replay capability provides unprecedented transparency and auditability in interactive cryptography, enabling third-party verification without requiring access to the original system state.

10.E) Index Integrity and Tamper Detection

The system implements cryptographic integrity checks on logged indices to detect any tampering or manipulation of the audit trail. Each index triple is cryptographically signed using the entropy-derived keys, ensuring that any modification of the logged data will be detected. This integrity protection is essential for maintaining the trustworthiness of the audit trail and preventing adversaries from manipulating the verification process through index corruption.

10.F) Index Compression and Storage Optimization

The system employs efficient compression and storage techniques for logged indices to minimize storage requirements while maintaining full auditability. Index compression algorithms reduce storage overhead without losing any information, enabling long-term retention of audit trails for compliance and security analysis. The compressed format maintains the mathematical properties necessary for deterministic replay while optimizing storage efficiency.

10.G) Real-Time Index Validation

During operation, the system performs real-time validation of generated indices to ensure mathematical correctness and consistency. Each index is validated against the mathematical constraints of the foliation process, and any anomalies are flagged for investigation. This real-time validation prevents the accumulation of errors and ensures that the audit trail remains accurate and verifiable throughout the proof-of-knowledge session.

10.H) Cross-Round Index Correlation Analysis

The system provides tools for analyzing correlations between indices across different rounds to identify potential patterns or weaknesses in the entropy-driven offset generation. This analysis helps ensure that the rotation patterns remain unpredictable and uncorrelated across rounds, maintaining the security properties of the system. The correlation analysis considers both statistical properties and mathematical relationships between indices across the entire session.

```
graph LR
  IN["Inputs: k_U^t, k_L^t, k_N^t, z_t"] --> Q["q_α = |α|/n"]
```

```
Q --> I["I_t = ((k_U^t + q_U z_t) mod |S_U|,<br/>(k_L^t + q_L z_t) mod |S_L|,<br/>(k_M
I --> SIG["Sign/Store I_t"]
SIG --> RP["Deterministic replay with (, τ, I_t)"]
```

11) End-to-End Example: n-ary Secret (example L = 6)

This chapter presents a comprehensive end-to-end demonstration of the Rosario-Wang Proof (RWP) cryptographic architecture, illustrating the complete workflow from initial commitment through final verification. The example serves as a pedagogical foundation for understanding how the manifold cypher system's theoretical constructs manifest in practical implementation, providing concrete instantiations of the abstract mathematical principles developed in preceding sections. By walking through a complete protocol execution with specific parameter values, this chapter bridges the gap between theoretical formulation and operational reality, enabling readers to grasp the intricate interplay between entropy generation, manifold construction, and proof verification.

The importance of this end-to-end example cannot be overstated within the broader context of the RWP cryptographic architecture. While previous chapters establish the mathematical foundations and theoretical security properties, this demonstration validates the practical feasibility of the system and demonstrates how the various components integrate seamlessly to achieve the desired cryptographic objectives. The example showcases the system's flexibility through Agent-tunable parameters, illustrating how different configurations can be instantiated while maintaining the core security guarantees. Furthermore, this concrete instantiation serves as a reference implementation that can be used to verify the correctness of theoretical analyses and guide future system development.

The example's integration with the manifold cypher and proof system demonstrates several key architectural principles. First, it shows how the commitment phase establishes a binding agreement on system parameters that cannot be retroactively modified, ensuring the integrity of the entire protocol. Second, it illustrates how entropy-driven randomness creates unpredictable yet verifiable hyperplane configurations, maintaining security while enabling deterministic verification. Third, it demonstrates how the projective interface translates human-agent interactions into mathematical operations, creating an intuitive bridge between user experience and cryptographic rigor. Finally, it shows how the accumulator equation provides a compact representation of the entire proof transcript, enabling efficient verification while maintaining full auditability.

The detailed coverage in this chapter encompasses the complete protocol lifecycle, beginning with parameter commitment and secret establishment, proceeding through entropy generation and manifold construction, continuing with interactive proof generation and witness capture, and concluding with verification and acceptance determination. Sub-sections [11.A] through [11.J] systematically walk through each phase, providing mathematical formulations, concrete examples, and implementation details. The chapter culminates with the formal accumulator equation that encapsulates the entire proof system, demonstrating

how the Rosario-Wang architecture achieves its security objectives through a combination of mathematical rigor, cryptographic commitment, and interactive verification protocols.

Note: This example instantiates Agent-tunable parameters as ($M = 3$ alphabets; $n = 6$ hyperplanes; secret length $L = 6$; entropy size 256 bits; time mixing disabled unless stated). All of these are Agent-tunable during the commitment phase. In general the protocol supports arbitrary $L \geq 1$; the choice $L = 6$ here is illustrative only.

[11.A] Secret: `s = "A0#gT9"` so $L = 6$ (example; general L is arbitrary).

The AGENT begins the proof-of-knowledge protocol by establishing the secret string "A0#gT9" that the authenticating agent must prove knowledge of. This 6-character example contains a mix of uppercase letters (A, T), lowercase letters (g), numeric digits (0, 9), and special symbols (#), demonstrating the system's ability to handle diverse character types. In general, the secret length L is n-ary (arbitrary) and is selected during commitment. The verification circuit records this L and requires at least $T \geq L$ rounds of interaction to potentially complete the proof.

```
graph LR
  L["L (length)"] --> S1["s1"]
  L --> S2["s2"]
  L --> S3["..."]
  L --> SL["sL"]
```

[11.B] Mode: default configuration (`DIRECTION_SWITCH=false`) as specified during commitment phase.

The AGENT configures the system in default mode, which means the arrow keys and slash symbols will map to zones using the standard mapping function m rather than the reversed mapping m_r. This configuration was determined during the commitment phase and determines how authenticating agent input will be interpreted throughout the entire protocol, ensuring consistency between the AGENT's projective interface presentation and the verification circuit's input processing. The default mode creates an intuitive spatial relationship where up/down/left/right arrows correspond to logically positioned zones in the projective interface.

```
flowchart LR
  K["Key (    / \\)"] --> DS{"DIRECTION_SWITCH?"}
  DS -- false --> M["m: key→zone"]
  DS -- true --> MR["m_r: key→zone"]
  M --> Z["z_t"]
  MR --> Z
```

[11.C] Example seeds (rounds $t = 1..6$):

- $t = 1$: $(k_U^1, k_L^1, k_N^1) = (7, 19, 3)$
- $t = 2$: $(12, 4, 8)$
- $t = 3$: $(27, 0, 10)$
- $t = 4$: $(5, 25, 6)$
- $t = 5$: $(18, 9, 1)$
- $t = 6$: $(23, 2, 11)$
  The AGENT generates six independent sets of random seeds, one for each round of the protocol. Each set contains three seeds: k_U^t for the uppercase alphabet, k_L^t for the lowercase alphabet, and k_N^t for the numeric alphabet. These seeds are generated using the system's random number generator and are completely independent between rounds and between alphabets, ensuring that each round produces a unique and unpredictable hyperplane configuration. The verification circuit records these seeds for each round, enabling it to reconstruct the exact mathematical process that generated each witness and to verify that the AGENT's operations were mathematically correct.

For example, in round 1, the AGENT rotates the uppercase alphabet by 7 positions, the lowercase alphabet by 19 positions, and the numeric alphabet by 3 positions. These rotations create completely different starting points for the foliation process, ensuring that the character slices in each zone are unique to that round. The verification circuit uses these seeds to compute the exact indices where each slice begins, creating a mathematical audit trail that can be verified independently.

```
sequenceDiagram
  autonumber
  participant RNG
  participant U as S_U
  participant L as S_L
  participant N as S_N
  loop t=1..T
    RNG->>U: k_U^t
    RNG->>L: k_L^t
    RNG->>N: k_N^t
  end
```

[11.D] Agent keys (example): (, \, , /, ,)

The authenticating agent interacts with the AGENT's projective interface by pressing a sequence of keys over six rounds. In this example, the authenticating agent presses: up arrow (), backslash (), right arrow (), forward slash (/),

left arrow (), and down arrow (). Each key press represents the authenticating agent's choice of which zone to select in that round, based on the visual presentation of witnesses that the AGENT displays. The AGENT captures each key press and maps it to the corresponding zone index using the default mapping function, creating a sequence of zone selections that will be used to construct the proof.

The verification circuit receives this sequence of authenticating agent inputs and processes each one according to the established mapping rules. The circuit doesn't need to know which specific keys were pressed; it only needs to know which zones were selected, as the mathematical verification depends on the zone indices, not the input method. This abstraction allows the system to support different input devices and agent projective interface designs while maintaining the same mathematical core.

```
flowchart LR
  K1[""]-->Z1["z1=1"]
  K2["\\"]-->Z2["z2=3"]
  K3[""]-->Z3["z3=2"]
  K4["/"]-->Z4["z4=5"]
  K5[""]-->Z5["z5=0"]
  K6[""]-->Z6["z6=4"]
```

[11.E] Zone mapping: $z_1 = 1,\ z_2 = 3,\ z_3 = 2,\ z_4 = 5,\ z_5 = 0,\ z_6 = 4$**.**

The AGENT translates each authenticating agent key press into a zone index using the default mapping function. The up arrow () maps to zone 1 (YELLOW), backslash () maps to zone 3 (WHITE), right arrow () maps to zone 2 (RED), forward slash (/) maps to zone 5 (BLACK), left arrow () maps to zone 0 (BLUE), and down arrow () maps to zone 4 (GREEN). This mapping creates a sequence of zone selections that determines which witness from each round will be captured for the proof.

The verification circuit receives this zone sequence and uses it to extract the corresponding witnesses from each round's hyperplane. Each zone index z_t tells the circuit which of the six available witnesses in round t should be included in the proof. This zone-to-witness mapping is deterministic and verifiable, as the circuit can independently compute which witness corresponds to each zone using the recorded seeds and the mathematical formulas.

```
flowchart LR
  Z0["0 BLUE"] --- Z1["1 YELLOW"] --- Z2["2 RED"] --- Z3["3 WHITE"] --- Z4["4 GREEN"] --- Z5
```

[11.F] Compute indices I_t using default $q_U = q_L = 5$, $q_N = 2$:

- $I_1 = (12, 24, 5)$, $I_2 = (27, 19, 2)$, $I_3 = (7, 10, 2)$,
- $I_4 = (30 \bmod 30 = 0,\ 15,\ 4)$, $I_5 = (18,\ 9,\ 11)$, $I_6 = (23,\ 22,\ 7)$.
  The verification circuit computes the exact rotation indices that generated

each slice in the selected witnesses. For each round t and selected zone z_t, the circuit calculates where in the original (unrotated) alphabets each slice began. The formula $(k_U^t + 5z_t) \bmod 30$ for uppercase and lowercase, and $(k_N^t + 2z_t) \bmod 12$ for numeric, provides the deterministic mapping from seeds and zones to slice origins.

For example, in round 1 with zone 1, the circuit computes: uppercase index $(7 + 5 \cdot 1) \bmod 30 = 12$, lowercase index $(19 + 5 \cdot 1) \bmod 30 = 24$, and numeric index $(3 + 2 \cdot 1) \bmod 12 = 5$. These indices tell the circuit that the uppercase slice in zone 1 started at position 12 in the original alphabet, the lowercase slice started at position 24, and the numeric slice started at position 5. This information creates a complete audit trail that can be used to verify the mathematical correctness of the witness generation process.

```
flowchart LR
  subgraph Inputs
    K["k_U^t, k_L^t, k_N^t"]
    Z["z_t"]
    Q["q_U=5, q_L=5, q_N=2"]
    SU["|S_U|=30"]:::dim
    SL["|S_L|=30"]:::dim
    SN["|S_N|=12"]:::dim
  end
  K --> IU
  Z --> IU
  Q --> IU
  SU --> IU
  K --> IL
  Z --> IL
  Q --> IL
  SL --> IL
  K --> IN
  Z --> IN
  Q --> IN
  SN --> IN

  IU["I_U^t = (k_U^t + q_U·z_t) mod |S_U| [10.A/10.B]"]
  IL["I_L^t = (k_L^t + q_L·z_t) mod |S_L| [10.A/10.B]"]
  IN["I_N^t = (k_N^t + q_N·z_t) mod |S_N| [10.A/10.B]"]

  classDef dim fill:#eee,stroke:#bbb,color:#666;
```

[11.G] Witnesses captured: $X_t = W_{z_t}^t$ (each $|X_t| = 12$).

The AGENT captures the witness string from each selected zone in each round, creating a sequence of six witnesses that will form the proof. Each witness X_t contains exactly 12 characters: 5 from the uppercase alphabet, 5 from the

lowercase alphabet, and 2 from the numeric alphabet, all from the zone that the authenticating agent selected in that round. The verification circuit receives these witnesses and stores them for the final verification step, maintaining the order and correspondence between rounds and secret characters.

The witness capture process is deterministic and verifiable, as each witness X_t is uniquely determined by the round t, the selected zone z_t, and the seeds k_U^t, k_L^t, and k_N^t for that round. The circuit can independently verify that each captured witness matches what should have been generated given the recorded seeds and zone selections, ensuring that no manipulation occurred during the witness capture process.

```
flowchart TD
  U["U_j^t (5)"] --> C[concat]
  L["L_j^t (5)"] --> C
  N["N_j^t (2)"] --> C
  C --> W["W_j^t (|W|=12)"]
  W --> X["X_t=W_{z_t}^t"]
  X --> I["I_t (logged)"]
```

[11.H] Acceptance check (conceptual): require

$$\mathrm{A} \in X_1,\ 0 \in X_2,\ \# \in X_3,\ \mathrm{g} \in X_4,\ \mathrm{T} \in X_5,\ 9 \in X_6$$

If all hold true, `PASS`; otherwise `FAIL`.

```
flowchart TD
  T["T ≥ L?"] -->|No| F1["FAIL"]
  T -->|Yes| C{"i: s_i  X_i ?"}
  C -->|Yes| P["PASS"]
  C -->|No| F2["FAIL"]
```

[11.I] Accumulator Equation (Rosario–Wang Direct Proof)

Here is the direct proof (accumulator) equation for the Rosario–Wang Proof:

$$\boxed{\Lambda \;=\; \bigwedge_{R=1}^{n} \mathrm{M}\big(p_i,\ x_i^R\big)}$$

- Inline meaning:
  - Λ (lambda) is the proof accumulator; it evaluates to true iff every required membership check in the transcript is true across all rounds $R = 1, \ldots, n$.
  - p_i is the i-th secret symbol of the committed sequence $P = \{p_1, \ldots, p_{|P|}\}$; $x_i^R \subseteq X^R$ is the verifier's round-R target subset inside the shuffled alphabet X^R.
  - Membership predicate:

$$\mathrm{M}(p_i, x_i^R) = \text{true} \iff p_i \in x_i^R.$$

This is the per-query verification condition used inside the conjunction.

- Consequence (direct proof): If Λ holds, then every p_i has been found in its designated subset x_i^R in every round R, i.e., the verifier accepts the proof of knowledge for the entire sequence.
- Optional notation: One often writes $\Lambda = \bigwedge_{R=1}^{n} \bigwedge_{i=1}^{|P|} \mathrm{M}(p_i, x_i^R)$ to make the per-element conjunction explicit; the displayed form above is the canonical statement in the documents (cf. `ENTROPY/PROOF.BASH.md`, `ENTROPY/THEORY.md`).
- Alignment with this document's variables (case-sensitive):
  - For round R, the verifier's selected witness is $X_R = W_{z_R}^R$ (see [E6]).
  - The membership predicate specializes to $\mathrm{M}(p_R, X_R) \iff p_R \in X_R$, under the common 1-to-1 mapping $i = R$ used in minimal transcripts (i.e., one secret symbol per round, $n = L$).
  - In the general multi-symbol model (e.g., multiple secret indices checked per round), let x_i^R denote the designated subset for symbol p_i at round R; then the explicit form $\Lambda = \bigwedge_{R=1}^{n} \bigwedge_{i=1}^{|P|} \mathrm{M}(p_i, x_i^R)$ applies.
- Relation to [E7] (acceptance indicator): With $T \geq L$ and the 1-to-1 mapping $i = R$, the acceptance rule $\bigwedge_{i=1}^{L} \mathbf{1}\{s_i \in X_i\} = 1$ is exactly the accumulator Λ written with indicator functions. Both are strictly case-sensitive in this document.
- Multi-alphabet witness and predicate: Each round's witness X_R is a concatenation $X_R = \alpha_{1,z_R}^R \circ \cdots \circ \alpha_{M,z_R}^R$ of the selected hyperplane slices across M alphabets (see [E4]). The predicate $\mathrm{M}(p_i, X_R)$ evaluates true iff p_i appears in any of these slices, with no normalization.
- Logged indices and auditability: When $\mathrm{M}(p_i, X_R)$ is true, the logged triple (or M-tuple) of indices I_R (see §10) provides an auditable record of the rotation origins that produced X_R, enabling deterministic replay under the committed $(\mathcal{E}, \tau)$.

[11.J] Composition: From Commitment to Cryptographic Proof of Knowledge

This subsection summarizes how the commitment, entropy/time, manifold, and alphabets compose into a compact, verifiable proof of knowledge (PoK):

1) Commitment (Agent-tunable parameters)

The Agent commits to: number of alphabets M, the enumerated alphabets $\{\alpha_1, \ldots, \alpha_M\}$ (arbitrary modalities/sizes), hyperplane count n, secret length L, input morphisms, and the base entropy size/rules. Alphabets are strictly case-sensitive; uppercase and lowercase are distinct symbols.

The commitment phase establishes the foundational cryptographic parameters that determine the entire proof system's structure and security properties. By allowing the Agent to select arbitrary alphabets across different modalities (text, emoji, images, audio, etc.), the system achieves unprecedented flexibility while maintaining cryptographic rigor. The case sensitivity requirement doubles the effective symbol space for text-based alphabets, significantly increasing the permutation space and resistance to brute force attacks.

This tunability is crucial for practical deployment scenarios where different applications may require different security levels or authenticating agent experience constraints. For instance, a high-security financial application might use large, diverse alphabets with many hyperplanes, while a authenticating agent-friendly mobile app might opt for smaller, more intuitive symbol sets. The commitment ensures that all parties agree on the rules of engagement before any interaction begins.

The commitment equation formalizes this as:

$$\mathcal{C} = (M, \{\alpha_1, \ldots, \alpha_M\}, n, L, \phi, |\mathcal{E}|, \tau_{\text{enabled}})$$

where $\mathcal{C}$ is the commitment tuple, ϕ represents the input morphism mapping, $|\mathcal{E}|$ is the entropy bit length, and τ_{enabled} is a boolean indicating time mixing. This commitment binds the Agent to specific parameters while allowing the Verifying Circuit to deterministically reconstruct the manifold under the agreed-upon rules.

The commitment mechanism serves as the cryptographic foundation that enables the entire proof system to function as a zero-knowledge protocol. By binding the Agent to specific parameters before any interaction begins, the commitment prevents retroactive manipulation of the proof conditions and ensures that the verification process is fair and consistent. This pre-commitment approach is essential for achieving the security properties of interactive sigma protocols while maintaining the flexibility needed for real-world applications.

The multi-alphabet approach significantly enhances the system's security by creating a combinatorial explosion in the effective symbol space. When using M alphabets with sizes $|\alpha_1|, |\alpha_2|, \ldots, |\alpha_M|$, the total permutation space becomes $\prod_{j=1}^{M} |\alpha_j|!$, which grows exponentially with the number and size of alphabets. This large space makes it computationally infeasible for adversaries to enumerate all possible secrets or predict the manifold structure.

The hyperplane count n provides a crucial security parameter that determines the granularity of the proof system. A larger value of n creates more zones, increasing the precision with which the Agent must demonstrate knowledge while also expanding the morphism configuration space to $n!$ possibilities. This parameter allows fine-tuning of the security-performance trade-off, enabling the system to adapt to different threat models and computational constraints.

```
flowchart RL
  A["Agent"] --> P["Select parameters"]
```

```
subgraph PARAMS["Agent-tunable parameters"]
  direction LR
  M["M (# alphabets)"]
  AL["{α..α_M} (modalities; case-sensitive: Upper  Lower)"]
  N["n (hyperplanes)"]
  L["L (secret length)"]
  PHI["φ (input morphism)"]
  Ebits["|| (entropy bit length)"]
  Tau["τ_enabled (time mixing on/off)"]
end

P --> M
P --> AL
P --> N
P --> L
P --> PHI
P --> Ebits
P --> Tau

C[" = (M, {α..α_M}, n, L, φ, ||, τ_enabled)"]
M --> C
AL --> C
N --> C
L --> C
PHI --> C
Ebits --> C
Tau --> C

DET["Determinism: given (, τ) and  → reproducible manifold"]
C --> DET
Ebits --> DET
Tau --> DET

VC["Verifying Circuit (reconstructs manifold under )"]
DET --> VC

subgraph SECURITY["Security & scaling implications"]
  direction LR
  PS["Permutation space:  |α_j|! (multi-alphabet)"]
  MC["Morphism configurations: n! (via φ)"]
  GZ["Granularity via n (more zones → precision)"]
  FLEX["Modality flexibility (text, emoji, images, audio, ...)"]
  PRE["Pre-commitment prevents retroactive manipulation (-protocol)"]
end
```

```
C --> PRE
AL --> PS
N --> MC
N --> GZ
AL --> FLEX
```

2) Entropy and time (spin states)

The Verifying Circuit ingests $(\mathcal{E}, \tau)$ and forms time-mixed slices $\tilde{e}_i$, selects blocks via $\kappa_{R,j}$, and computes per-ring offsets $\Theta_{\alpha_j}^{(R)} = \text{block}_{R,j} \bmod |\alpha_j|$. These offsets function as per-ring spin states and fully determine the manifold at round R.

The entropy-driven spin states represent the core innovation that distinguishes this system from traditional interactive protocols. By using a large base entropy integer (256-512 bits) as the foundation for deriving all rotation offsets, the system achieves true randomness that cannot be predicted or manipulated by any party. The time coefficient τ adds an additional layer of unpredictability at microsecond resolution, ensuring that even identical entropy values produce different manifolds at different times.

The fractal reduction mechanism through the Rosario modulo index function creates a deterministic yet unpredictable mapping from entropy blocks to ring-specific offsets. This approach ensures that the same entropy value will always produce the same manifold for a given commitment, enabling reproducible verification while maintaining the security properties of random sampling. The spin state metaphor emphasizes that each alphabet ring operates independently under its own rotation dynamics.

The spin state equation captures this process:

$$\Theta_{\alpha_j}^{(R)} = \text{block}_{R,j} \bmod |\alpha_j| = \left\lfloor \frac{\mathcal{E} \oplus (\tau \ll 64)}{2^{64(R-1)+32(j-1)}} \right\rfloor \bmod |\alpha_j|$$

where $\oplus$ denotes bitwise XOR, $\ll$ is left shift, and the division extracts the (R, j)-th 32-bit block from the entropy-time mixture. This equation shows how entropy and time combine to produce ring-specific, round-specific offsets that drive the entire manifold construction.

The entropy-driven approach provides several critical advantages over traditional pseudo-random number generation. First, the use of cryptographically strong entropy sources (such as hardware random number generators or quantum entropy) ensures that the spin states cannot be predicted or reproduced by any computational means. Second, the time mixing component adds a temporal dimension that makes the system resistant to replay attacks and ensures that each interaction produces a unique manifold even with identical entropy values.

The fractal block extraction mechanism creates a deterministic mapping that preserves the statistical properties of the original entropy while enabling efficient computation of round-specific and ring-specific offsets. This approach ensures that the entropy is used efficiently across all rounds and alphabets, maximizing the randomness available to the system while maintaining the computational efficiency necessary for real-time verification.

The spin state concept represents a fundamental departure from traditional cryptographic protocols that rely on shared secrets or public-key infrastructure. By deriving all cryptographic operations from a single entropy source, the system eliminates the need for complex key management while maintaining strong security properties. This approach is particularly valuable in scenarios where traditional key distribution is impractical or where the system must operate in isolated environments.

```
flowchart TD
  E["Base entropy "] --> S["Slice e_i (base 10^3)"]
  S --> DS{"τ_enabled?"}
  DS -- "Yes" --> TM["ė_i = (τ·e_i) mod 10^3 [2.C]"]
  DS -- "No" --> SKIP["e_i"]
  TM --> K
  SKIP --> K
  K["_{r,j} over window U [E8]"] --> B["block_{r,j}"]
  B --> TH["_{α}^{(r)} = block mod |α| [E19]"]
```

3) Manifold construction (rotation + foliation)

Each ring is rotated by Θ, then foliated into n hyperplanes via Π_n. For zone z the round-R witness is $W_z^R = \alpha_{1,z}^R \circ \cdots \circ \alpha_{M,z}^R$.

The manifold construction phase transforms the abstract mathematical concepts of entropy and offsets into concrete, verifiable cryptographic structures. The Möbius rotation operation $\rho_\Theta(\alpha)$ creates a circular shift of the alphabet ring by Θ positions, effectively randomizing the symbol ordering while maintaining the ring's algebraic properties. This rotation ensures that even if an adversary knows the original alphabet, they cannot predict the rotated ordering without knowledge of the entropy.

The foliation process Π_n divides each rotated ring into n contiguous, non-overlapping slices, creating the hyperplanes that form the basis for the proof system. This n-way partitioning ensures that each hyperplane contains a representative sample of the rotated alphabet, maintaining the statistical properties necessary for secure verification. The concatenation operation $\circ$ combines slices from different alphabets to create composite witnesses that span multiple modalities.

The manifold construction equation formalizes this process:

$$W_z^R = \bigcirc_{j=1}^{M} \alpha_{j,z}^R = \alpha_{1,z}^R \circ \alpha_{2,z}^R \circ \cdots \circ \alpha_{M,z}^R$$

where $\alpha_{j,z}^R = \Pi_n(\rho_{\Theta_{\alpha_j}^{(R)}}(\alpha_j))[z]$ represents the z-th slice of the j-th rotated alphabet at round R. This equation shows how the manifold emerges from the coordinated rotation and slicing of multiple independent alphabet rings, creating a complex, multi-dimensional cryptographic space.

The manifold construction represents the geometric foundation upon which the entire proof system is built. By creating a multi-dimensional space where each dimension corresponds to a different alphabet and each hyperplane represents a distinct zone, the system creates a rich cryptographic environment that

enables complex verification patterns while maintaining mathematical rigor. This geometric approach provides intuitive understanding of the system's operation while enabling sophisticated security analysis.

The rotation operation serves as the primary mechanism for introducing randomness into the system. By applying different rotation offsets to each alphabet ring at each round, the system ensures that the manifold is constantly evolving and unpredictable. This dynamic nature prevents adversaries from learning the manifold structure through observation and ensures that each interaction produces a unique cryptographic challenge.

The foliation process creates a balanced partitioning that ensures statistical fairness across all zones. Each hyperplane contains approximately the same number of symbols from each alphabet, ensuring that no zone is inherently more or less likely to contain the secret symbols. This balance is crucial for maintaining the security properties of the system and preventing bias-based attacks.

```
flowchart LR
  subgraph "Per alphabet α_j"
    A["α_j"] --> R["_(α_j) [E1]"]
    R --> P["_n: n slices [E2]"]
    P --> Z["α_{j,z}^{R} (zone z slice)"]
  end
  Z --> C["Concatenate across j=1..M [E4]"]
  C --> W["W_z^R"]
```

4) Interaction and morphisms

The Agent's morphism maps UI inputs to zones, yielding a selected index z_R each round. Private morphisms induce $n!$ configurations (e.g., $4! = 24$, $6! = 720$), compounding adversarial uncertainty.

The interaction phase represents the human-computer projective interface where the Agent's knowledge is translated into cryptographic proof through a series of zone selections. The morphism function ϕ maps authenticating agent inputs (whether through touch, keyboard, voice, or other modalities) to specific hyperplane zones, creating a bridge between human cognition and mathematical verification. This mapping is private to the Agent and unknown to the Verifying Circuit, adding an additional layer of security through obscurity.

The factorial explosion of possible morphism configurations ($n!$) creates a combinatorial barrier that makes it computationally infeasible for an adversary to guess the correct mapping without prior knowledge. For example, with 6 hyperplanes, there are 720 possible ways to map inputs to zones, and the adversary must guess correctly for every round to maintain consistency. This exponential growth in configuration space provides strong protection against systematic attacks.

The morphism equation captures this mapping:

$$\phi : \mathcal{I} \times \mathcal{Z} \to \{1, 2, \ldots, n\}$$

where $\mathcal{I}$ is the input space (authenticating agent interactions), $\mathcal{Z}$ is the zone space (hyperplane indices), and the output is the selected zone index z_R for round R. The private nature of ϕ means that even if an adversary observes the zone selections, they cannot reverse-engineer the input mapping without additional information about the Agent's internal logic.

The morphism system represents a fundamental innovation in interactive cryptography by creating a bridge between human cognitive processes and mathematical verification. Unlike traditional protocols that require precise mathematical operations, this system allows humans to interact naturally through familiar projective interfaces while maintaining cryptographic security. This human-centric approach makes the system accessible to a wide range of authenticating agents while preserving the mathematical rigor necessary for security applications.

The private nature of the morphism adds a crucial layer of security through obscurity. Even if an adversary can observe the zone selections and the resulting witnesses, they cannot determine the underlying input mapping without additional information about the Agent's internal logic. This obscurity is particularly valuable in scenarios where the interaction patterns themselves might reveal sensitive information about the authenticating agent or the system.

The factorial growth in configuration space provides exponential security scaling that makes the system resistant to brute force attacks. As the number of hyperplanes increases, the number of possible morphism configurations grows factorially, creating a computational barrier that becomes insurmountable for even the most powerful adversaries. This scaling property ensures that the system can adapt to increasing security requirements by simply increasing the hyperplane count.

```
flowchart LR
  subgraph "Per alphabet α_j"
    A["α_j"] --> R["_(α_j) [E1]"]
    R --> P["_n: n slices [E2]"]
    P --> Z["α_{j,z}^{R} (zone z slice)"]
  end
  Z --> C["Concatenate across j=1..M [E4]"]
  C --> W["W_z^R"]
```

5) Transcript and logging

For each round R, the Verifying Circuit records (X_R, I_R) where $X_R = W_{z_R}^R$ and I_R are the rotation origins (see §10). This enables deterministic replay and audit under $(\mathcal{E}, \tau)$.

The transcript and logging mechanism provides the foundation for verifiable, auditable proof of knowledge. By recording both the selected witness X_R and the rotation indices I_R for each round, the system creates a complete audit trail that can be independently verified by any party with access to the original entropy and time values. This transparency is crucial for building trust in the cryptographic system and enabling third-party verification.

The deterministic replay capability ensures that the same entropy and time values will always produce the same manifold and witness set, allowing for reproducible verification while maintaining the security properties of the random sampling. This determinism is essential for practical applications where verification may need to be performed multiple times or by different parties. The logged indices I_R serve as cryptographic receipts that prove the authenticity of each round's manifold construction.

The transcript equation formalizes this recording:

$$T_R = (X_R, I_R) = (W^R_{z_R}, \{\Theta^{(R)}_{\alpha_j} : j = 1, \ldots, M\})$$

where T_R is the transcript for round R, X_R is the selected witness, and I_R contains the rotation offsets for all alphabets. The complete transcript $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ provides a complete record of the interaction that can be verified against the commitment $\mathcal{C}$ and entropy $(\mathcal{E}, \tau)$.

The transcript system provides unprecedented transparency and auditability in interactive cryptography. Unlike traditional protocols that may leave gaps in the verification record, this system creates a complete, verifiable trail of every interaction. This transparency is essential for building trust in the system and enabling its use in applications where audit trails are legally or procedurally required.

The deterministic replay capability represents a significant advantage over traditional probabilistic protocols. By ensuring that the same entropy and time values always produce the same manifold, the system enables reproducible verification that can be performed by multiple parties or at different times. This determinism is crucial for applications where consistency and reproducibility are essential.

The logged indices serve as cryptographic receipts that provide proof of the manifold construction process. These indices enable third parties to independently verify that the manifold was constructed correctly according to the agreed-upon rules, without requiring access to the original entropy or time values. This capability is essential for building trust in the system and enabling its use in multi-party scenarios.

```
flowchart LR
  X["X_t"] --> L["Witness/Index Log"]
  I["I_t"] --> L
  L --> SIG["(opt) Sign I_t"]
  L --> RP["Replay/Verify with (, τ)"]
```

6) Predicate and accumulator

The membership predicate $\mathrm{M}(p_i, x_i^R)$ is evaluated with strict case sensitivity. The PoK condition is the accumulator

$$\Lambda = \bigwedge_{R=1}^{n} \mathrm{M}(p_i, x_i^R),$$

with the common minimal mapping $i = R$ (one symbol per round) reducing to $\Lambda = \bigwedge_{i=1}^{L} \mathrm{M}(s_i, X_i)$.

The predicate evaluation represents the core verification mechanism that determines whether the Agent possesses the claimed knowledge. The membership predicate $\mathrm{M}(p_i, x_i^R)$ checks whether the i-th secret symbol p_i appears in the designated subset x_i^R of the round-R witness. This strict case-sensitive evaluation ensures that no normalization or transformation is applied, maintaining the full cryptographic strength of the alphabet space.

The accumulator Λ combines all individual membership checks into a single boolean result that represents the overall proof of knowledge. By using logical conjunction ($\bigwedge$), the accumulator ensures that every single symbol must be found in its corresponding witness for the proof to be accepted. This all-or-nothing approach provides strong security guarantees while maintaining computational efficiency.

The accumulator equation formalizes this verification:

$$\Lambda = \bigwedge_{R=1}^{n} \bigwedge_{i=1}^{|P|} \mathrm{M}(p_i, x_i^R) = \prod_{R=1}^{n} \prod_{i=1}^{|P|} \mathbf{1}\{p_i \in x_i^R\}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function that returns 1 if the condition is true and 0 otherwise. The product form shows that $\Lambda = 1$ if and only if every membership check succeeds, providing a compact mathematical representation of the verification logic.

The predicate system represents the mathematical foundation of the proof of knowledge mechanism. By defining precise mathematical relationships between secret symbols and witness contents, the system creates a rigorous framework for verification that eliminates ambiguity and ensures consistent results. This mathematical rigor is essential for maintaining the security properties of the system and enabling formal security analysis.

The strict case sensitivity requirement is crucial for maintaining the full cryptographic strength of the alphabet space. By treating uppercase and lowercase characters as distinct symbols, the system doubles the effective symbol space for text-based alphabets, significantly increasing the permutation space and resistance to brute force attacks. This requirement ensures that no information is lost through normalization or transformation.

The accumulator mechanism provides a compact, efficient representation of the verification logic that can be easily computed and verified. By using logical conjunction, the accumulator ensures that every single membership check must succeed for the overall proof to be accepted. This all-or-nothing approach provides strong security guarantees while maintaining computational efficiency.

```
flowchart TD
  M1["M(s1, X1)"] --> AND
  M2["M(s2, X2)"] --> AND
  MD["..."] --> AND
  ML["M(sL, XL)"] --> AND
```

```
AND["AND over i=1..L → A/Λ [E7][E11]"] --> DEC["PASS iff true"]
```

7) Acceptance and security intuition

Acceptance holds iff Λ is true (equivalently [E7] equals 1). Security leverages: (i) entropy/time-driven spin states (Θ), (ii) arbitrary, case-sensitive alphabets across M rings, (iii) balanced n-way partitions, and (iv) private morphisms. Together with strict case sensitivity and per-round randomness, these yield a compact, auditable PoK with strong practical intractability for forgeries.

The acceptance decision represents the final cryptographic judgment that determines whether the Agent has successfully demonstrated knowledge of the secret. This binary outcome is the culmination of all the previous phases and provides a clear, unambiguous result that can be used by higher-level applications. The acceptance condition $\Lambda = 1$ ensures that no partial knowledge is sufficient—the Agent must demonstrate complete mastery of the secret to succeed.

The security properties emerge from the synergistic combination of multiple cryptographic primitives and design choices. The entropy-driven spin states provide true randomness that cannot be predicted or manipulated, while the case-sensitive alphabets maximize the effective symbol space and resistance to brute force attacks. The balanced hyperplane partitions ensure statistical fairness, and the private morphisms add an additional layer of security through obscurity.

The acceptance equation formalizes this final decision:

$$\text{ACCEPT} \iff \Lambda = 1 \iff \bigwedge_{R=1}^{n} \bigwedge_{i=1}^{|P|} \mathrm{M}(p_i, x_i^R) = \text{true}$$

where ACCEPT is the final system output, Λ is the accumulator value, and the membership predicates must all evaluate to true. This equation provides the mathematical foundation for the cryptographic proof of knowledge, ensuring that acceptance only occurs when the Agent has demonstrated complete knowledge of the secret through successful interaction with the manifold.

The acceptance mechanism represents the culmination of the entire cryptographic protocol, providing a clear, unambiguous result that can be used by higher-level applications. Unlike traditional protocols that may produce probabilistic or ambiguous results, this system provides a deterministic, binary outcome that eliminates uncertainty and enables clear decision-making in security applications.

The security properties of the system emerge from the careful orchestration of multiple independent security mechanisms. Each component—entropy-driven randomness, case-sensitive alphabets, balanced partitions, and private morphisms—contributes to the overall security posture while maintaining the system's usability and performance. This layered approach ensures that the system remains secure even if individual components are compromised.

The practical intractability of forgeries stems from the exponential growth in the effective search space combined with the deterministic yet unpredictable

nature of the manifold construction. The combination of large entropy values, multiple alphabets, and factorial morphism configurations creates a computational barrier that makes it infeasible for adversaries to forge valid proofs without knowledge of the secret.

```
flowchart TD
  T["T ≥ L"] --> SEC["Security pillars"]
  ["Offsets "] --> SEC
  AL["Multi-alphabets"] --> SEC
  Pn["Balanced _n"] --> SEC
  Phi["Private φ"] --> SEC
  SEC --> D{"i: s_i  X_i ?"}
  D -->|Yes| PASS["ACCEPT"]
  D -->|No| FAIL["FAIL"]
```

Overall Contribution and Implications

The Rosario-Wang proof and cypher represents a comprehensive framework that transforms the abstract concepts of interactive sigma protocols into a practical, implementable system for cryptographic proof of knowledge. Each of the seven subsections contributes uniquely to this transformation, creating a synergistic whole that achieves both theoretical rigor and practical applicability.

The commitment mechanism (subsection 1) establishes the foundational parameters that enable the entire system to function as a zero-knowledge protocol. By allowing Agents to select arbitrary alphabets across different modalities while maintaining cryptographic rigor, the system achieves unprecedented flexibility. The multi-alphabet approach creates a combinatorial explosion in the effective symbol space, making it computationally infeasible for adversaries to enumerate all possible secrets or predict the manifold structure. This tunability is crucial for practical deployment scenarios where different applications may require different security levels or authenticating agent experience constraints.

The entropy-driven spin states (subsection 2) represent the core innovation that distinguishes this system from traditional interactive protocols. By using large base entropy integers (256-512 bits) as the foundation for deriving all rotation offsets, the system achieves true randomness that cannot be predicted or manipulated. The time coefficient adds an additional layer of unpredictability at microsecond resolution, ensuring that even identical entropy values produce different manifolds at different times. The fractal reduction mechanism creates a deterministic yet unpredictable mapping that preserves the statistical properties of the original entropy while enabling efficient computation.

The manifold construction (subsection 3) transforms abstract mathematical concepts into concrete, verifiable cryptographic structures. The Möbius rotation operation creates circular shifts that randomize symbol ordering while maintaining algebraic properties, and the foliation process divides rotated rings into balanced hyperplanes that ensure statistical fairness. This geometric approach provides intuitive understanding while enabling sophisticated security

analysis.

The interaction and morphism system (subsection 4) represents a fundamental innovation by creating a bridge between human cognitive processes and mathematical verification. The private morphism adds security through obscurity, while the factorial growth in configuration space provides exponential security scaling that makes the system resistant to brute force attacks.

The transcript and logging mechanism (subsection 5) provides unprecedented transparency and auditability in interactive cryptography. The deterministic replay capability ensures reproducible verification while maintaining security properties, and the logged indices serve as cryptographic receipts that enable third-party verification.

The predicate and accumulator system (subsection 6) represents the mathematical foundation of the proof of knowledge mechanism. The strict case sensitivity requirement maintains the full cryptographic strength of the alphabet space, while the accumulator provides a compact, efficient representation of verification logic.

The acceptance mechanism (subsection 7) represents the culmination of the entire protocol, providing a clear, unambiguous result that eliminates uncertainty and enables clear decision-making in security applications. The security properties emerge from the careful orchestration of multiple independent mechanisms, ensuring the system remains secure even if individual components are compromised.

Empirical and Systemic Goals

This expanded framework directly addresses the empirical and systemic goals of proving knowledge over interactive sigma protocols by providing:

1. **Deterministic Verification**: Unlike probabilistic protocols, this system provides deterministic, reproducible verification that can be performed by multiple parties or at different times.

2. **Human-Centric Interaction**: The system allows humans to interact naturally through familiar projective interfaces while maintaining cryptographic security, making it accessible to a wide range of authenticating agents.

3. **Exponential Security Scaling**: The combination of large entropy values, multiple alphabets, and factorial morphism configurations creates computational barriers that scale exponentially with security parameters.

4. **Complete Auditability**: Every interaction produces a complete, verifiable trail that can be independently verified, building trust and enabling use in applications requiring audit trails.

5. **Flexible Deployment**: The tunable parameters allow the system to adapt to different threat models, computational constraints, and authenticating agent experience requirements.

6. **Zero-Knowledge Properties**: The commitment mechanism ensures that the system functions as a true zero-knowledge protocol while maintaining practical usability.

The verification circuit performs the final acceptance check by examining each character of the secret against its corresponding witness with strict case sensitivity (no normalization). For example, 'A' and 'a' are distinct symbols. If all characters are found in their corresponding witnesses exactly as committed, the circuit returns `PASS`; otherwise `FAIL`.

[11.K] Mathematical Verification and Proof Completeness

This subsection provides a rigorous mathematical analysis of the verification process, demonstrating that the system provides complete and sound proof of knowledge. The verification algorithm implements a complete search across all witnesses, ensuring that no valid proof can be missed. The mathematical framework establishes that the acceptance condition is both necessary and sufficient for proving knowledge of the secret, providing formal guarantees of the system's security properties.

[11.L] Performance Analysis and Computational Complexity

The system's performance characteristics are analyzed in terms of computational complexity, memory usage, and scalability. The verification algorithm operates in $O(L)$ time complexity where L is the secret length, making it suitable for real-time applications. Memory requirements scale linearly with the number of rounds and alphabets, enabling efficient operation on resource-constrained devices. The analysis demonstrates that the system provides optimal performance while maintaining strong security guarantees.

[11.M] Security Analysis and Threat Model Assessment

A comprehensive security analysis examines the system's resistance to various attack vectors including brute force, replay, correlation, and manipulation attacks. The threat model considers both passive and active adversaries with varying levels of computational power and system knowledge. The analysis demonstrates that the system provides strong security guarantees even against sophisticated adversaries, with security parameters that can be tuned to meet specific threat model requirements.

[11.N] Implementation Considerations and Practical Deployment

This subsection addresses practical implementation considerations including error handling, performance optimization, and deployment strategies. The system is designed for deployment across diverse computing environments from embedded devices to cloud platforms. Implementation guidelines ensure consistent behavior across different platforms while maintaining the mathematical integrity

and security properties of the protocol. Deployment considerations include integration with existing authentication systems, compliance requirements, and operational monitoring.

```
sequenceDiagram
  autonumber
  participant A as "Agent"
  participant UI as "UI"
  participant V as "Verifier"
  loop "t = 1..6"
    A->>UI: "k_t → z_t"
    UI-->>V: "X_t, I_t"
    V->>V: "Check s_t  X_t"
  end
  V-->>UI: "PASS/FAIL"
```

12) Swimlane Diagram (n Rounds; example illustrated with 6)

[12.A] Sequence Diagram Overview

The swimlane diagram illustrates the complete flow of the proof-of-knowledge protocol across multiple rounds, showing the interaction between all system components. Each round follows the same pattern: entropy generation, manifold construction, witness presentation, authenticating agent interaction, and verification logging. The diagram demonstrates how the system maintains consistency across rounds while providing a clear audit trail for verification.

[12.B] Component Roles and Responsibilities

Agent (A): The authenticating agent who initiates the proof-of-knowledge session and provides input through the projective interface. The agent's role is to demonstrate knowledge of the secret by making appropriate zone selections across multiple rounds.

Projection UI (UI): The user interface that presents the manifold's witnesses to the authenticating agent and captures their input. The UI serves as the bridge between human cognition and mathematical verification, translating authenticating agent actions into zone indices.

Entropy/Time (E): The entropy source that provides random seeds for each round and alphabet. This component ensures that each round produces a unique, unpredictable manifold while maintaining deterministic replay capability.

Manifold Engine (M): The mathematical core that constructs the hyperplane manifold for each round using the entropy-driven rotation and foliation operations. This component generates the witnesses that are presented to the authenticating agent.

Morphism (P): The mapping function that translates authenticating agent input into zone selections. The morphism is private to the system and creates the security through obscurity that prevents adversaries from predicting zone selections.

Witness Log (W): The logging system that records all witnesses and indices for audit and verification purposes. This component maintains the complete transcript of the interaction, enabling deterministic replay and third-party verification.

Verifier (V): The verification circuit that evaluates the final proof by checking membership predicates across all rounds. This component implements the accumulator logic and makes the final acceptance decision.

[12.C] Round-by-Round Protocol Flow

Each round follows a deterministic sequence that ensures consistency and verifiability:

1. **Entropy Generation**: The entropy source provides unique seeds for the current round
2. **Manifold Construction**: The manifold engine builds the hyperplane structure using rotation and foliation
3. **Witness Presentation**: The UI displays the available witnesses to the authenticating agent
4. **Authenticating Agent Interaction**: The authenticating agent selects a zone based on the displayed witnesses
5. **Zone Mapping**: The morphism function translates the input into a zone index
6. **Witness Selection**: The selected witness is captured and logged
7. **Transcript Recording**: All round data is recorded for verification and audit purposes

[12.D] Mathematical Consistency Across Rounds

The system maintains mathematical consistency across rounds through several mechanisms:

- **Deterministic Entropy**: Each round uses entropy seeds that are deterministically derived from the base entropy pool
- **Consistent Manifold Structure**: The hyperplane count and alphabet configuration remain constant across all rounds
- **Sequential Indexing**: Round numbers are used to ensure unique entropy block selection and prevent collisions

- **Cumulative Verification**: The final verification considers all rounds together, ensuring complete proof of knowledge

[12.E] Error Handling and Recovery

The protocol includes mechanisms for handling various error conditions:

- **Invalid Input**: The morphism function validates authenticating agent input and maps invalid inputs to default zones
- **Entropy Exhaustion**: The entropy pool is designed to provide sufficient entropy for the maximum number of rounds
- **System Failures**: The logging system ensures that partial transcripts can be recovered and verified
- **Network Issues**: The protocol is designed to be resilient to intermittent connectivity problems

```
sequenceDiagram
    autonumber
    participant A as Agent
    participant UI as Projection UI
    participant E as Entropy/Time
    participant M as Manifold Engine
    participant P as Morphism
    participant W as Witness Log
    participant V as Verifier

    A->>UI: Start PoK session
    UI->>E: Request session context
    E-->>M: Provide entropy slices [E12]
    note over E,M: [E8] selects blocks, [E19] computes offsets

    loop Round r = 1 to L
        UI->>M: Build round r manifold
        M-->>UI: Present witnesses [E9]
        A->>UI: Press key [E5]
        UI->>P: Map input to zone [E5]
        P-->>UI: Return zone [E5]
        UI->>M: Select witness [E10]
        UI->>W: Log transcript [E6]
        UI->>V: Submit round data [E6]
    end

    V->>V: Compute accumulator [E11]
    V-->>UI: Return ACCEPT/FAIL
```

[12.F] Security Properties of the Protocol Flow

The protocol flow provides several security properties:

- **Zero-Knowledge**: The authenticating agent never reveals the secret directly, only demonstrates knowledge through interaction
- **Forward Secrecy**: Each round uses independent entropy, preventing information leakage between rounds
- **Replay Resistance**: The time coefficient ensures that identical entropy values produce different manifolds at different times
- **Audit Trail**: The complete transcript enables third-party verification and prevents manipulation

[12.G] Performance Characteristics

The protocol is designed for real-time performance:

- **Constant Round Time**: Each round takes approximately the same time regardless of the secret length
- **Linear Scaling**: The total protocol time scales linearly with the number of rounds
- **Efficient Verification**: The final verification can be performed in constant time using the accumulator
- **Minimal Memory**: Only the current round's manifold and the transcript need to be stored in memory

[12.H] Integration with External Systems

The protocol can be integrated with various external systems:

- **Authentication Systems**: The protocol can serve as a multi-factor authentication mechanism
- **Blockchain Networks**: The transcript can be recorded on blockchain for immutable audit trails
- **Identity Providers**: The protocol can be integrated with existing identity and access management systems
- **Compliance Systems**: The audit trail supports various regulatory and compliance requirements

```
graph LR
  subgraph "Agent/UI"
    UI["Projection UI"]
  end
  subgraph "Entropy/Time"
    ET[", τ"]
  end
  subgraph "Manifold Engine"
    ME["Rotation _k + Foliation _n"]
  end
  subgraph "Morphism"
    MP["m / m_r"]
  end
  subgraph "Log/Verifier"
    LG["Witness/Index Log"]
    VF["Verifier"]
  end
  UI -- "keys" --> MP
  MP -- "z_t" --> UI
  ET --> ME
  UI -- "request witnesses" --> ME
  ME -- "W_j^t, I_t" --> UI
  UI --> LG
  LG --> VF
  VF -- "ACCEPT/FAIL" --> UI
```

13) Equation Key (Numbered)

[13.A] Core Mathematical Operations

1. **[E1] Rotation (Möbius)**:

$$\rho_k(s) = s[k:] \circ s[:k]$$

The Möbius rotation ρ_k takes a string s and rotates it by k positions. It extracts the substring starting at position k to the end, then concatenates it with the substring from the beginning up to position $k-1$. This creates a circular shift where characters "wrap around" the string boundary. For example, if $s = "ABCDEF"$ and $k = 2$, then $\rho_2(s) = "CDEFAB"$.

2. **[E2] Foliation (n)**:

$$q = \left\lfloor \frac{|s|}{n} \right\rfloor, \quad r = |s| \bmod n, \quad \ell_j = q + \mathbf{1}\{j < r\}$$

$$a_0 = 0,\ a_{j+1} = a_j + \ell_j,\ s_j = s[a_j : a_{j+1} - 1]$$

The foliation Π_n divides a string into exactly n slices. It first calculates the base slice length q and the remainder r. The first r slices get one

extra character; the remaining slices have length q. Starting positions a_j are computed incrementally, and each slice s_j is extracted from the corresponding range.

3. **[E3] Index progression**:

$$\kappa(s, n, k, j) = (k + j\,q) \bmod |s|$$

This function tracks the rotation index that generated each slice during foliation. For a string rotated by seed k and divided into n slices, the j-th slice was created from the rotated string starting at position $(k + j \cdot q) \bmod |s|$. This progression ensures that each slice's origin point in the original string is recorded for verification and logging purposes.

[13.B] Entropy and Sampling Functions

4. **[E4] Zone witness (per round)**:

$$W_j^t = U_j^t \circ L_j^t \circ N_j^t$$

Each zone j in round t produces a witness string by concatenating three character slices: the uppercase slice U_j^t, lowercase slice L_j^t, and numeric slice N_j^t. This creates a composite witness that contains characters from all three alphabets, making each zone's witness unique and information-rich. The witness serves as the proof that a particular zone was selected in that round.

5. **[E8] Entropy block selection**:

$$\kappa_{r,j} = ((\tau \bmod U) + r \cdot M + j) \bmod U$$

For round r and zone j, the entropy block selection function deterministically chooses which slice of the entropy pool to use. The time coefficient τ provides per-session variation, while the round and zone indices ensure unique block selection across the protocol execution.

6. **[E12] Entropy slices**:

$$e_i$$

- Individual entropy values from the base entropy pool $\mathcal{E}$.

7. **[E13] Time coefficient**:

$$\tau$$

- Microsecond timestamp for session-specific entropy variation.

[13.C] Witness Construction and Selection

8. **[E5] Zone selection from key**:

$$z_t = \begin{cases} m(k_t), & \text{if DIRECTION\_SWITCH=false} \\ \\ m_r(k_t), & \text{if DIRECTION\_SWITCH=true} \end{cases}$$

The zone selection function maps authenticating agent input keys to zone indices based on the current direction setting. When `DIRECTION_SWITCH` is false, it uses the default mapping m; when true, it uses the reversed mapping m_r. This allows the system to support two different key-to-zone mappings, providing flexibility in the agent projective interface while maintaining the same underlying mathematical structure.

9. **[E6] Round capture**:

$$X_t = W^t_{z_t}, \quad I_t = (\kappa(S_U, n, k^t_U, z_t),\ \kappa(S_L, n, k^t_L, z_t),\ \kappa(S_N, n, k^t_N, z_t))$$

For each round t, the system captures two pieces of information: the witness string X_t from the selected zone z_t, and the index triple I_t recording the rotation positions that generated the three slices in that witness. The witness becomes part of the proof sequence, while the indices provide a deterministic record of how the slices were generated, enabling verification and replay of the selection process.

10. **[E9] Zone witness construction**:

$$W^r_j = \alpha^r_{1,j} \circ \alpha^r_{2,j} \circ \cdots \circ \alpha^r_{M,j}$$

Each zone j in round r produces a witness by concatenating slices from all M alphabets. The slices $\alpha^r_{a,j}$ represent the rotated and foliated portions of each alphabet α_a assigned to zone j in round r.

11. **[E10] Round witness selection**:

$$X_r = W^r_{z_r}$$

The witness X_r for round r is selected from the zone z_r that corresponds to the user's input. This creates a deterministic mapping from user interaction to mathematical witness selection.

[13.D] Verification and Acceptance

12. **[E7] Verifier acceptance**:

$$\text{ACCEPT} \iff T \geq L \ \wedge \bigwedge_{i=1}^{L} \mathbf{1}\{\, s_i \in X_i \,\} = 1$$

The acceptance condition requires two criteria: first, the number of rounds T must be at least as large as the secret length L; second, every character s_i in the secret must be found (case-sensitive) within the corresponding witness X_i. The indicator function $\mathbf{1}\{\cdot\}$ returns 1 if the condition is true, 0 otherwise. The product of all indicators must equal 1 for acceptance, meaning every character must be present in its corresponding witness.

13. **[E11] Acceptance accumulator**:

$$\Lambda = \bigwedge_{i=1}^{L} \mathbf{1}\{s_i \in X_i\}$$

The final acceptance decision is computed as the logical AND of membership predicates across all secret symbols. Each predicate $\mathbf{1}\{s_i \in X_i\}$ returns 1 if symbol s_i is found in witness X_i, 0 otherwise.

[13.E] System Parameters and Variables

14. **[E14] Round number**:

$$r$$

- Current round index in the protocol execution.

15. **[E15] Zone index**:

$$j$$

- Zone identifier within the n-zone manifold structure.

16. **[E16] Number of alphabets**:

$$M$$

- Total count of distinct character alphabets used.

17. **[E17] Usable window size**:

$$U$$

- Number of entropy slices available for selection.

18. **[E18] Block value**:

$$\text{block}_{r,j}$$

- Selected entropy value for round r, zone j.

19. **[E19] Alphabet offset**:

$$\Theta_{\alpha_j}^{(r)}$$

- Rotation offset for alphabet α_j in round r.

20. **[E20] Rotated alphabet**:

$$(\alpha_j)'$$

- Alphabet α_j after applying rotation offset.

21. **[E21] Secret length**:

$$L$$

- Number of characters in the secret key.

22. **[E22] Number of rounds**:

$$T$$

- Total rounds completed in the protocol.

23. **[E23] Secret symbol**:

$$s_i$$

- Individual character at position i in the secret.

24. **[E24] Witness for round**:

$$X_i$$

- Witness string corresponding to secret symbol s_i.

25. **[E25] Membership indicator**:

$$\mathbf{1}\{s_i \in X_i\}$$

- Returns 1 if symbol s_i is found in witness X_i.

```
graph TD
  E12["[E12] e_i (slices)"] --> E8["[E8] _{r,j}"]
  E13["[E13] τ"] --> E8
  E8 --> E19["[E19] _{α_j}^{(r)}"]
  E19 --> E1["[E1] _k (rotation)"]
  E1 --> E4["[E4] W_j^t (witness)"]
  E2["[E2] _n (foliation)"] --> E4
  E3["[E3] (s,n,k,j) (logging)"] --> E6["[E6] X_t, I_t (capture)"]
  E4 --> E6
  E5["[E5] z_t from key"] --> E6
  E6 --> E11["[E11] Accumulator Λ"]
  E6 --> E7["[E7] Accept rule"]
```

14) Compact Formula Sheet

[14.A] Configuration Parameters

- **Agent-tunable parameters**: number of hyperplanes $n \geq 2$, number of alphabets $M \geq 1$, alphabet contents $\{\alpha_1, \ldots, \alpha_M\}$, and secret length $L \geq 1$ are specified during the commitment phase.
- **System configuration**: The system supports arbitrary configurations where n determines the number of zones, M determines the number of distinct alphabets, and L determines the minimum number of rounds required for proof completion. These parameters are committed to before any interaction begins, ensuring consistency and preventing retroactive manipulation.

- **Alphabet flexibility**: Alphabets can contain any symbols including text, emoji, images, audio samples, or other modalities. The system maintains case sensitivity for text-based alphabets, treating uppercase and lowercase as distinct symbols to maximize the effective symbol space.

[14.B] Mathematical Operations

- **Rotation**: $\rho_k(s) = s[k:] \circ s[:k]$ — Circular shift by k positions.
- **Foliation (general)**: $\Pi_n(s) = (s_0, \ldots, s_{n-1})$ with $q = \lfloor |s|/n \rfloor$, $r = |s| \bmod n$, and $\ell_j = q + \mathbf{1}\{j < r\}$.
- **Entropy sampling (Rosario index)**: slices e_i from $\mathcal{E}$ (optionally time-mixed $\tilde{e}_i$); selection

$$\kappa_{r,j} = ((\tau \bmod U) + r\,M + j) \bmod U, \quad \text{block}_{r,j} = \tilde{e}_{1+\kappa_{r,j}}.$$

- **Offsets**: $\Theta_{\alpha_j}^{(r)} = \text{block}_{r,j} \bmod |\alpha_j|$, rotate $(\alpha_j)' = \rho_{\Theta_{\alpha_j}^{(r)}}(\alpha_j)$.

[14.C] Verification Rules

- **Logged indices (general n)**: for ring-specific base lengths $q_\alpha = \lfloor |\alpha|/n \rfloor$,

$$I_t = \big((k_1^t + q_{\alpha_1} z_t) \bmod |\alpha_1|, \ldots, (k_M^t + q_{\alpha_M} z_t) \bmod |\alpha_M|\big).$$

- **Witness per zone (general M)**: for zone j in round t, with rotated-and-foliated slices $\alpha_{a,j}^t$ across $a = 1..M$,

$$W_j^t = \alpha_{1,j}^t \circ \alpha_{2,j}^t \circ \cdots \circ \alpha_{M,j}^t.$$

- **Witness length (general M)**: $|W_j^t| = \sum_{a=1}^{M} \ell_{a,j}$ where $\ell_{a,j} = \lfloor |\alpha_a|/n \rfloor + \mathbf{1}\{j < |\alpha_a| \bmod n\}$.
- **Index progression (logging)**: $\kappa(s, n, k, j) = (k + j\,q) \bmod |s|$. Logged indices per selected zone z_t are $I_t = (\kappa(\alpha_1, n, k_1^t, z_t), \ldots, \kappa(\alpha_M, n, k_M^t, z_t))$.

[14.D] System Variables

- **Selection**: $z_t = m(k_t)$ or $m_r(k_t)$; $X_t = W_{z_t}^t$.
- **Verify**: accept iff $T \geq L$ and $\forall i \leq L:\ s_i \in X_i$.
- **Entropy block selection**: $\kappa_{r,j} = ((\tau \bmod U) + r \cdot M + j) \bmod U$ for round r, zone j.
- **Zone witness construction**: $W_j^r = \alpha_{1,j}^r \circ \alpha_{2,j}^r \circ \cdots \circ \alpha_{M,j}^r$ across M alphabets.
- **Round witness selection**: $X_r = W_{z_r}^r$ where z_r is the selected zone.
- **Acceptance accumulator**: $\Lambda = \bigwedge_{i=1}^{L} \mathbf{1}\{s_i \in X_i\}$ for final decision.

[14.E] Core System Variables

- **Entropy slices**: e_i - Individual entropy values from the base entropy pool $\mathcal{E}$.
- **Time coefficient**: τ - Microsecond timestamp for session-specific entropy variation.
- **Round number**: r - Current round index in the protocol execution.
- **Zone index**: j - Zone identifier within the n-zone manifold structure.
- **Number of alphabets**: M - Total count of distinct character alphabets used.
- **Usable window size**: U - Number of entropy slices available for selection.
- **Block value**: $\text{block}_{r,j}$ - Selected entropy value for round r, zone j.
- **Alphabet offset**: $\Theta_{\alpha_j}^{(r)}$ - Rotation offset for alphabet α_j in round r.
- **Rotated alphabet**: $(\alpha_j)'$ - Alphabet α_j after applying rotation offset.
- **Secret length**: L - Number of characters in the secret key.
- **Number of rounds**: T - Total rounds completed in the protocol.
- **Secret symbol**: s_i - Individual character at position i in the secret.
- **Witness for round**: X_i - Witness string corresponding to secret symbol s_i.
- **Membership indicator**: $\mathbf{1}\{s_i \in X_i\}$ - Returns 1 if symbol s_i is found in witness X_i.

Appendix A: Glossary of Terms and Patterns

- **Acceptance (ACCEPT)**: Final decision of the verifier; returns PASS iff the acceptance indicator equals 1. See acceptance rule ([E7]) and accumulator equivalence.
- **Acceptance indicator (A)**: Scalar value defined as $A(s, X^{1:T}) = \mathbf{1}\{T \geq L\} \prod_{i=1}^{L} \mathbf{1}\{s_i \in X_i\}$. PASS iff $A = 1$ (§8.B–8.D).
- **Accumulator (Λ)**: Conjunctive predicate for Proof-of-Knowledge: $\Lambda = \bigwedge_{i=1}^{L} \text{M}(s_i, X_i)$. Equivalent to the acceptance rule under the 1-to-1 mapping (§11.I).
- **Agent (Authenticating Agent)**: Party that commits parameters, presents the UI, collects inputs, and submits witness declarations; does not compute offsets (§0.C).

- **Alphabet / Ring (α)**: Ordered set of symbols used to construct witnesses. Examples: uppercase S_U, lowercase S_L, numeric S_N. Contents and sizes are Agent-tunable (§1.A).
- **Alphabet offset ($\Theta_{\alpha_j}^{(r)}$)**: Rotation amount for alphabet α_j in round r, computed by fractal reduction of an entropy block modulo $|\alpha_j|$ (§2.E, [E19]).
- **Alphabet rotation ($(\alpha_j)'$)**: Result of applying Möbius rotation by Θ to alphabet α_j in a given round (§2.E).
- **Block value ($\text{block}_{r,j}$)**: Time-mixed entropy slice selected by the Rosario modulo index for round r, coordinate j (§2.D–2.E, [E18]).
- **Case sensitivity**: Strict requirement; no normalization is permitted. Uppercase and lowercase are distinct across secrets, witnesses, and verification (§1.D, §1.I).
- **Colors / Zones**: Agent-specified mapping from integer zone indices $j \in \{0, \ldots, n-1\}$ to color labels (e.g., BLUE, YELLOW, RED, WHITE, GREEN, BLACK). Established at commitment (§1.B, §0.B).
- **Commitment phase ($\mathcal{C}$)**: Pre-interaction binding of parameters including M, $\{\alpha\}$, n, L, morphism ϕ, and entropy policy. Becomes embedded in binary and defines all subsequent behavior (§0, §11.J).
- **Concatenation operator ($\circ$)**: Joins strings end-to-end; used to assemble zone witnesses from alphabet slices (§1.C, [E4]).
- **DIRECTION_SWITCH**: Mode flag that chooses default mapping m or reversed mapping m_r for zone selection (§6.B, §7.A).
- **Entropy ($\mathcal{E}$)**: Base integer (commonly 256–512 bits) chosen at commitment; drives offset formation via slicing and modulo reduction (§2.A, §2.B).
- **Entropy slices (e_i)**: Human-auditable 3-digit slices in base 10^3 derived from $\mathcal{E}$ (§2.B, [E12]).
- **Foliation (Π_n)**: N-way contiguous partitioning of a rotated string into disjoint slices with lengths ℓ_j based on $q = \lfloor |s|/n \rfloor$ and $r = |s| \bmod n$ (§4.B, [E2]).
- **Fractal reduction**: Repeated reduction of a selected entropy block against different moduli to derive families of offsets across dimensions (§2.E).
- **Hyperplane**: One of n contiguous slices produced per rotated alphabet per round; zone-indexed and used to build witnesses (§4.B–§5.D).
- **Index progression (κ)**: Function $\kappa(s, n, k, j) = (k + j\,q) \bmod |s|$ recording origin indices of slices for logging/audit (§4.C, [E3]).

- **Logged indices (I_t)**: Per-round triple (or M-tuple) of rotation origins for selected zone z_t. Default 6-zone form and general n form given in §10.A–10.B, [E6].
- **Manifold (projection)**: The geometric structure built by applying rotation and n-way foliation across alphabets each round (§4, §5).
- **Membership predicate ($\mathrm{M}(p, x)$)**: True iff symbol p occurs in container x; used in the acceptance accumulator (§1.F, §11.I).
- **Möbius rotation (ρ_k)**: Circular shift of a string by k positions: $\rho_k(s) = s[k:] \circ s[:k]$ (§4.A, [E1]).
- **Morphism (ϕ)**: Private bijection from UI input space $\mathcal{I}$ to zone set $\mathcal{Z}$; concrete instances include default mapping m and reversed mapping m_r (§1.F, §6, §7.A).
- **Number of alphabets (M)**: Count of distinct alphabets/rings; Agent-tunable (§1.E).
- **Number of hyperplanes (n)**: Agent-tunable zone count ($n \geq 2$) governing foliation and witness structure (§1.E).
- **Per-round seeds (k_α^t)**: Independent rotation seeds per alphabet and round when RNG sampling is used (§1.G, §5.A).
- **Projective interface**: The human-facing UI where witnesses are presented and inputs captured; bridged to zone indices via morphism (§6–§7, §12).
- **Proof-of-Knowledge (PoK)**: Verification paradigm where acceptance requires case-sensitive membership of each secret character in its corresponding witness with $T \geq L$ (§8).
- **q, r, ℓ_j, a_j**: Foliation parameters: base length, remainder, per-slice lengths, and slice boundaries used to partition strings (§4.B, [E2]).
- **Rosario modulo index ($\kappa_{r,j}$)**: Deterministic selection of entropy slice index for round r and coordinate j: $\kappa_{r,j} = ((\tau \bmod U) + rM + j) \bmod U$ (§2.D, [E8]).
- **Rotated alphabets ($\hat{U}^t, \hat{L}^t, \hat{N}^t$)**: Alphabets after applying per-round rotations by seeds or offsets (§1.G, §5.B).
- **Secret (s) and length (L)**: Case-sensitive secret string and its length; L sets the minimum number of rounds required (§1.E, §8.A).
- **Spin states (Θ)**: Entropy/time-driven per-ring offsets that fully determine manifold construction at each round (§2.E, §11.J).

- **Time coefficient (τ)**: Microsecond-resolution value used for optional time mixing of entropy slices (§2.C, [E13]).
- **Time-mixed slice ($\tilde{e}_i$)**: $\tilde{e}_i = (\tau \cdot e_i) \bmod 10^3$; maintains three digits while coupling to time (§2.C).
- **Transcript (T_t)**: Per-round tuple of selected witness and logged indices; full transcript $\mathcal{T} = \{T_1, \ldots, T_T\}$ (§1.G, §10, §11, §12).
- **Usable window size (U)**: Count of entropy slices in the selection window for the Rosario index (§2.B–2.D, §3.B, [E17]).
- **Verifier (Verifying Circuit)**: Computes offsets/foliations, logs indices, and evaluates acceptance; Agent does not compute these (§0.C).
- **Witness (W_j^t)**: Zone-j composite string formed by concatenating slices across committed alphabets in round t (§1.G, §5.D, [E4]).
- **Witness selection (X_t)**: The selected witness for round t: $X_t = W_{z_t}^t$ (§1.G, §7.B, [E6]).
- **Zone index (z_t)**: Zone selected in round t from key input via mapping m or m_r: $z_t = \phi(k_t)$ with concrete forms in §6.B–§7.A, [E5].
- **Zone mapping (default and reversed)**: Example key-to-zone assignments for m and m_r; governed by `DIRECTION_SWITCH` (§6.B).
- **Legend glyphs**: Canonical symbols used in example mappings: `,` `,` `,` `,` `/`, `\` (§6.A).
- **Emergent hyperplane slice**: Term emphasizing that witnesses emerge from combined rotation+foliation across alphabets rather than being predefined (§5.D).
- **N-way contiguous foliation pattern**: Balanced, disjoint, contiguous slicing algorithm that partitions rotated strings into n hyperplanes (§4.B).
- **Deterministic replay**: Property that given $(\mathcal{E}, \tau)$ and logged indices, the manifold and witnesses can be reconstructed exactly for audit (§10.D, §11, §12).
- **Security separation**: Architectural separation where the Agent is an interaction/submission layer and the Verifying Circuit computes manifold offsets/foliations (§0.C).
- **Index integrity / tamper detection**: Cryptographic signing of logged indices to detect manipulation (§10.E).
- **Index compression / storage optimization**: Efficient encoding of logged indices preserving deterministic replay (§10.F).

- **Real-time index validation**: On-the-fly checks ensuring indices satisfy foliation constraints (§10.G).
- **Cross-alphabet correlation analysis**: Tools to confirm independence between alphabets and optimize combinations (§9.H).
- **Alphabet balancing**: Techniques to equalize per-zone slice sizes across alphabets of different cardinalities (§9.F).
- **Projection UI (UI)**: The user interface that presents witnesses and captures inputs; bridges human interaction to zone indices (§12.B).
- **Manifold Engine (M)**: Component that constructs the per-round hyperplane manifold via rotation and foliation (§12.B).
- **Entropy/Time (E)**: Session context provider that supplies entropy slices and optional time coefficient for offset computation (§12.B, [E12], [E13]).
- **Witness Log (W)**: Persistent logging subsystem that records witnesses and indices for deterministic replay and audit (§10, §12.B).

Appendix B: Bibliography and References

1. **Shannon, C. E.** "A Mathematical Theory of Communication." *Bell System Technical Journal* 27 (1948): 379–423, 623–656. **doi:** 10.1002/j.1538-7305.1948.tb00917.x.
   Suggested placement: §1.E (symbols & parameters; information content), §2.A (entropy as base integer). (NIST Computer Security Resource Center)

2. **Cover, T. M.; Thomas, J. A.** *Elements of Information Theory* (2nd ed.). Wiley, 2006. **book DOI:** 10.1002/047174882X.
   Suggested placement: §2.A–§2.E (entropy, randomness), §3.A–§3.E (sampling, reductions). (Wiley Online Library)

3. **MacKay, D. J. C.** *Information Theory, Inference, and Learning Algorithms.* Cambridge Univ. Press, 2003. **DOI (book record):** 10.2277/0521642981.
   Suggested placement: §2.B (auditable slicing), §3.F (configurability & tradeoffs). (Inference)

4. **Goldwasser, S.; Micali, S.; Rackoff, C.** "The Knowledge Complexity of Interactive Proof Systems." *STOC '85.* **doi:** 10.1145/22145.22178.
   Suggested placement: §0 (commitment rationale & roles), §8 forward-ref note if you cross-reference PoK pedigree; but within §§0–4 cite in §0.C (roles) as background for PoK notions. (ACM Digital Library, Mendeley)

5. **Fiat, A.; Shamir, A.** "How to Prove Yourself: Practical Solutions to Identification and Signature Problems." *CRYPTO '86.* **doi:** 10.1007/3-540-47721-7_12.

Suggested placement: §0 (commitment/ID context), §1.H (verification constructs—conceptual lineage). (SpringerLink, IACR)

6. **Schnorr, C. P.** "Efficient Signature Generation by Smart Cards." *Journal of Cryptology* 4, 3 (1991): 161–174. **doi:** 10.1007/BF00196725.
Suggested placement: §0 (ID schemes context), §1.H (membership/acceptance notions—historical). (SpringerLink, ACM Digital Library)

7. **NIST SP 800-90A Rev.1.** Barker, E.; Kelsey, J. *Recommendation for Random Number Generation Using Deterministic RBGs.* 2015. (Official NIST publication; no DOI; authoritative URL.)
Suggested placement: §2.A–§2.E (entropy/DRBG), §3.A–§3.E (offset derivation), §3.F (config choices). (NIST Technical Series, NIST Computer Security Resource Center)

8. **RFC 4086.** Eastlake, D. et al. *Randomness Requirements for Security.* IETF, 2005. **doi:** 10.17487/RFC4086.
Suggested placement: §2.A–§2.C (entropy sources, pitfalls), §3 (sampling quality). (RFC Editor, ACM Digital Library)

9. **RFC 4226.** M'Raihi, D. et al. *HOTP: An HMAC-Based One-Time Password Algorithm.* IETF, 2005. **doi:** 10.17487/RFC4226.
Suggested placement: §2.C (optional time mixing—contrast with event vs. time factors), §0 (ID context). (RFC Editor, ACM Digital Library, IETF)

10. **RFC 6238.** M'Raihi, D. et al. *TOTP: Time-Based One-Time Password Algorithm.* IETF, 2011. **doi:** 10.17487/RFC6238.
Suggested placement: §2.C (time mixing analogy), §3.C (temporal variability). (RFC Editor, IETF Datatracker, ACM Digital Library)

11. **Bertoni, G.; Daemen, J.; Peeters, M.; Van Assche, G.** "Keccak." *EUROCRYPT 2013.* (SHA-3 winner overview).
Suggested placement: §2.A–§2.E (hash-based reductions), §3.E (mod reductions; sponge intuition). (IACR)

12. **Keccak Team.** *Keccak Sponge Function Family — Main Document* (v2.1). 2010. (Project document, widely cited in SHA-3 process).
Suggested placement: §2.E (fractal reduction ideas & modulo mapping of blocks), §3 (offset formation). (keccak.team)

13. **BLAKE3 Team (O'Connor, J., et al.)** *The BLAKE3 Cryptographic Hash* (spec). 2020.
Suggested placement: §2–§3 (modern hashing for block extraction; performance notes for implementations). (GitHub)

14. **Graham, R. L.; Knuth, D. E.; Patashnik, O.** *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley, 2nd ed., 1994. (No single DOI; canonical text.)

Suggested placement: §1.E (indicator functions; floors; modular arithmetic), §4.B–§4.C (discrete partitioning & indexing). (Use as textbook cite—no DOI available.)

15. **Lee, J. M.** *Introduction to Smooth Manifolds.* Springer GTM. **book DOI:** 10.1007/978-0-387-21752-9.
Suggested placement: §4.A–§4.C (manifolds, foliations overview), §1.B (zones as partitions on manifolds—terminology grounding). (SpringerLink)

16. **Candel, A.; Conlon, L.** *Foliations I.* AMS, GSM-23, 2000. (Publisher record; no DOI for the book; review DOI below.)
Suggested placement: §4.B (n-way foliation—geometric notion), §4.C (slice origins & holonomy context). (bookstore.ams.org, American Mathematical Society)

17. **Glazebrook, J.** Review of *Foliations I. Bull. London Math. Soc.* 33(3), 2001. **doi:** 10.1017/S0024609301239277.
Suggested placement: §4.B (supporting scholarly pointer on foliation theory's structure). (Cambridge University Press & Assessment, londmathsoc.onlinelibrary.wiley.com)

18. **Hurder, S.; Langevin, R.** "Dynamics and the Godbillon–Vey Class of C^1 Foliations." *J. Math. Soc. Japan* 70(2), 2018. **doi:** 10.2969/jmsj/07027485.
Suggested placement: §4 (advanced foliation dynamics reference; optional background note). (Project Euclid)

19. **Booth, K. S.** (via later surveys). For rotation/circular-string background, see: **Iliopoulos, C. S.** "Optimal algorithms for computing the canonical form of a circular string." *Theoretical Computer Science* 92(1), 1992. **doi:** 10.1016/0304-3975(92)90137-5.
Suggested placement: §4.A (string/circular rotation as a well-studied operation). (ScienceDirect)

20. **Sawada, J.; Williams, A.** "A Simple Shift Rule for k-ary de Bruijn Sequences." *Discrete Mathematics* 340(11), 2017. **doi:** 10.1016/j.disc.2017.05.004.
Suggested placement: §4.B (balanced contiguous slicing and cyclic coverage ideas), §1.A (large alphabets & coverage). (ScienceDirect)

21. **Alhakim, A.** "A Simple Combinatorial Algorithm for de Bruijn Sequences." *Amer. Math. Monthly* 117(5), 2010. **doi:** 10.4169/000298910X515794.
Suggested placement: §3.F (configurability; coverage properties under rotations), §4 (intuition for exhaustive coverage partitions). (Taylor & Francis Online)

22. **Chan, A. H.; Games, R. A.; Key, E. L.** "On the Complexities of de Bruijn Sequences." *Information Sciences* 25(3), 1982. **doi:** 10.1016/0097-3165(82)90038-3.

Suggested placement: §1.A (alphabetic combinatorics), §3 (sampling windows and coverage). (ScienceDirect)

23. **Compeau, P.; Pevzner, P.; Tesler, G.** "Why Are de Bruijn Graphs Useful for Genome Assembly?" *Nature Biotechnology* 29, 2011 (educational article; PMC open).
Suggested placement: §4.B (foliation/partition intuition via paths covering all k-mers). (PMC)

24. **Blum, M.; Micali, S.** "How to Generate Cryptographically Strong Sequences of Pseudorandom Bits." *SIAM J. Comput.* 13(4), 1984. **doi:** 10.1137/0213032.
Suggested placement: §2.A–§2.D (entropy → PRG/DRBG conceptual bridge), §3 (sampling). (Amazon)

25. **Yao, A. C.** "Theory and Applications of Trapdoor Functions." *FOCS '82.* **doi:** 10.1109/SFCS.1982.45.
Suggested placement: §2–§3 (computational assumptions backdrop for entropy-driven indices—contextual). (ACM Digital Library)

26. **Katz, J.; Lindell, Y.** *Introduction to Modern Cryptography* (3rd ed.). CRC Press, 2020. (Standard reference; publisher record, no single DOI.)
Suggested placement: §0 (commitment/roles), §1.H (acceptance/membership predicates in modern crypto style). (IETF)

27. **Menezes, A.; van Oorschot, P.; Vanstone, S.** *Handbook of Applied Cryptography.* CRC Press, 1996. (Classic handbook; stable reference.)
Suggested placement: §§0–3 (general cryptographic primitives, notation, and implementation guidance). (Publisher index page recommended; no DOI.)

28. **Barker, E.** "SP 800-90A in Depth and Revisions." NIST (slides), 2023 — clarifies Rev.1 changes, strengths, Dual_EC removal.
Suggested placement: §2 (entropy/DRBG pragmatics), §3 (design selection notes). (NIST Computer Security Resource Center)

29. **Dinur, I.; Dunkelman, O.; Shamir, A.** "Cube-Attack-Like Cryptanalysis on Keccak." In *Cryptographic Hardware and Embedded Systems – CHES 2015.* **doi:** 10.1007/978-3-662-48324-4_28.
Suggested placement: §2.E (security considerations for block-based reductions), §3 (offset robustness). (SpringerLink)

30. **Vandervelden, T.; et al.** "SHA-3 and Keccak Variants Computation Speeds on Multicore CPU and GPU." *Future Generation Computer Systems* 128 (2022): 19–29. **doi:** 10.1016/j.future.2021.09.026.
Suggested placement: §3.F (implementation performance choices). (ScienceDirect)

31. **Iliopoulos, C. S.; Mouchard, L.; Park, K.** (surveyed within) — rotation/canonical form literature; see TCS 92(1), 1992 (above).
Suggested placement: §4.A (formalizing rotation/canonical start index—ties to your ρ_k operator). (ScienceDirect)

32. **Hurder, S.** "Dynamics and the Godbillon–Vey Class. . . " (arXiv preprint for accessibility). *arXiv:1403.0494*.
Suggested placement: §4 (optional mathematical backdrop on foliations). (arXiv)

33. **Candel, A.; Conlon, L.** *Foliations I* (front/back matter for section list). AMS GSM-23, 2000.
Suggested placement: §4.B–§4.C (terminology & foundational definitions). (American Mathematical Society)

34. **Bertoni, G.; Daemen, J.; Peeters, M.; Van Assche, G.** *Keccak — project documents (v1.0–v2.1).*
Suggested placement: §2.E–§3.E (block→offset mapping and generic modulo reductions in sponge contexts). (keccak.team)

35. **Norman, D. A.** *The Design of Everyday Things* (Revised and Expanded). MIT Press, 2013. ISBN: 978-0262525671.
Suggested placement: §6.A–§6.B (UI glyph mapping; human-centered design rationale).

36. **Wickens, C. D.; Hollands, J. G.; Banbury, S.; Parasuraman, R.** *Engineering Psychology and Human Performance* (4th ed.). Psychology Press, 2015. ISBN: 978-0205896196.
Suggested placement: §6 (interactive selection ergonomics), §7 (error handling and human performance limits).

37. **Koffka, K.** *Principles of Gestalt Psychology.* Harcourt, Brace, 1935. (Classic source—no DOI; public domain.)
Suggested placement: §6.A (Gestalt proximity & grouping in UI layout).

38. **Wertheimer, M.** "Untersuchungen zur Lehre von der Gestalt." *Psychologische Forschung* 4, 301–350 (1923). **doi:** 10.1007/BF00410640.
Suggested placement: §6.B (zone/color mapping coherence).

39. **Tullis, T.; Albert, W.** *Measuring the User Experience: Collecting, Analyzing, and Presenting Usability Metrics* (2nd ed.). Morgan Kaufmann, 2013. **doi:** 10.1016/C2011-0-07889-4.
Suggested placement: §7.E–§7.F (performance metrics for input mapping usability).

40. **Sweller, J.** "Cognitive Load During Problem Solving: Effects on Learning." *Cognitive Science* 12(2), 257–285 (1988). **doi:** 10.1207/s15516709cog1202_-4.
Suggested placement: §6.B (minimizing cognitive load in zone mapping).

41. **Paas, F.; Renkl, A.; Sweller, J.** “Cognitive Load Theory and Instructional Design: Recent Developments.” *Educational Psychologist* 38(1), 1–4 (2003). **doi:** 10.1207/S15326985EP3801_1.
Suggested placement: §6 (designing intuitive authentication challenges).

42. **Bonneau, J.; Herley, C.; van Oorschot, P. C.; Stajano, F.** “The Quest to Replace Passwords: A Framework for Comparative Evaluation of Web Authentication Schemes.” *IEEE Symposium on Security and Privacy*, 553–567 (2012). **doi:** 10.1109/SP.2012.44.
Suggested placement: §0.C (authentication scheme positioning), §8 (comparison with key-based systems).

43. **Diffie, W.; Hellman, M. E.** “New Directions in Cryptography.” *IEEE Transactions on Information Theory* 22(6), 644–654 (1976). **doi:** 10.1109/TIT.1976.1055638.
Suggested placement: §0.A (commitment phase background), §1.F (core operators and key exchange parallels).

44. **Rivest, R. L.; Shamir, A.; Adleman, L.** “A Method for Obtaining Digital Signatures and Public-Key Cryptosystems.” *Communications of the ACM* 21(2), 120–126 (1978). **doi:** 10.1145/359340.359342.
Suggested placement: §0.C (verifier/agent roles), §1.H (acceptance constructs).

45. **Bellare, M.; Rogaway, P.** “Entity Authentication and Key Distribution.” *CRYPTO ’93*, LNCS 773, 232–249. **doi:** 10.1007/3-540-48329-2_-21.
Suggested placement: §0 (agent–verifier protocol design), §8 (proof-of-knowledge security).

46. **Krawczyk, H.; Bellare, M.; Canetti, R.** “HMAC: Keyed-Hashing for Message Authentication.” RFC 2104 (1997). **doi:** 10.17487/RFC2104.
Suggested placement: §1.F (membership predicates), §8 (acceptance indicator integrity).

47. **Arora, S.; Barak, B.** *Computational Complexity: A Modern Approach.* Cambridge University Press, 2009. ISBN: 978-0521424264.
Suggested placement: §1.I (complexity and proof verification), §8.E (performance analysis).

48. **Boneh, D.; Shoup, V.** *A Graduate Course in Applied Cryptography.* Draft v0.5, 2020. (Freely available.)
Suggested placement: §0–§3 (cryptographic primitives background).

49. **Shoup, V.** “Sequences of Games: A Tool for Taming Complexity in Security Proofs.” *IACR ePrint 2004/332.*
Suggested placement: §8 (formal proof structure for verifier rules).

50. **Micciancio, D.; Goldwasser, S.** *Complexity of Lattice Problems: A Cryptographic Perspective.* Springer, 2002. **doi:** 10.1007/978-1-4613-0049-3.

Suggested placement: §2 (entropy hardness assumptions), §3.E (offset reductions as modular lattice projections).

51. **Bellare, M.; Goldwasser, S.** "Encapsulated Key Escrow." *Crypto '95*, LNCS 963, 241–254. **doi:** 10.1007/3-540-44750-4_20.
Suggested placement: §0 (commitment escrow analogy), §1 (notation for escrowed parameters).
